(15 pts) **1.** Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1].$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \Rightarrow \sum_{n=-\infty}^{\infty} e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1] e^{-j\omega n}$$

$$\Rightarrow \sum_{n=1}^{\infty} e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n e^{-j\omega n} \Rightarrow \sum_{n=1}^{\infty} e^{j\frac{\pi}{17}n - j\omega n}\left(\frac{1}{3}\right)^n$$

$$\Rightarrow \sum_{n=1}^{\infty} e^{jn(\frac{\pi}{17}-\omega)}\left(\frac{1}{3}\right)^n \Rightarrow \sum_{m=0}^{\infty} e^{j(m+1)(\frac{\pi}{17}-\omega)}\left(\frac{1}{3}\right)^{m+1}$$

$m = n-1$

$n = m+1$

$$\Rightarrow \frac{e^{j(\frac{\pi}{17}-\omega)}}{3} \sum_{m=0}^{\infty} e^{jm(\frac{\pi}{17}-\omega)}\left(\frac{1}{3}\right)^m$$

$$= \frac{e^{j(\frac{\pi}{17}-\omega)}}{3} \sum_{m=0}^{\infty}\left(\frac{1}{3}e^{j(\frac{\pi}{17}-\omega)}\right)^m$$

$$\boxed{= \frac{\frac{1}{3}e^{j(\frac{\pi}{17}-\omega)}}{1-\frac{1}{3}e^{j(\frac{\pi}{17}-\omega)}}}$$

(15 pts) **2.** Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$\mathcal{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi).$$

$$x(t) = \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{X}(\omega) e^{j\omega n} d\omega \Rightarrow \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi) e^{j\omega n} d\omega$$

$$\Rightarrow \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta(\omega - k\pi) e^{j\omega n} d\omega$$

$$\Rightarrow \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \int_{0}^{2\pi} \delta(\omega - k\pi) e^{j\omega n} d\omega$$

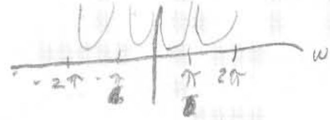
$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{jk\pi n} \Rightarrow \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{j\pi n}\right)^k$$

$$\boxed{= \frac{\frac{1}{2\pi}}{1 - \frac{1}{2} e^{j\pi n}}}$$

(15 pts) **3.** Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

$x[n] = 2j\left(e^{jg[n]} + e^{-jg[n]}\right)$



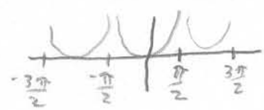
$\mathcal{X}(\omega)$ is pure imaginary and odd $\rightarrow$ $x[n]$ must be real and odd

$\rightarrow\leftarrow$

$x[n]$ is imaginary

$\boxed{\text{No}}$

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{3}{\cos\omega}$. Could Alice be right? Explain.



$\mathcal{X}(\omega)$ is real and even

$\therefore$ $x[n]$ must be real and even.

$\rightarrow\leftarrow$

$x[n]$ is imaginary

$\boxed{\text{No}}$

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{(\omega^2+1)^2}$. Could Devin be right? Explain.

$\mathcal{X}(\omega)$ not periodic

the F.T. of a D.T. signal is always periodic.

$\rightarrow\leftarrow$

$\boxed{\text{No}}$

4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

by (33) and (34)

$$\Rightarrow Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

Frequency Response

$\Rightarrow Y[n] = h[n] * X[n]$

by (40) and (33)

$$Y(\omega) = H(\omega)X(\omega) \Rightarrow \frac{Y(\omega)}{X(\omega)} = H(\omega)$$

$$Y(\omega)\left(-\frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega} + 1\right) = 2X(\omega)$$

$$\boxed{H(\omega) = \frac{2}{\left(-\frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega} + 1\right)}}$$

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$$h[n] = \mathcal{F}^{-1}(H(\omega)) \quad \frac{2}{\left(\frac{1}{8}(e^{-j\omega})^2 - \frac{3}{4}e^{-j\omega} + 1\right)} = \frac{A}{e^{-j\omega}-2} + \frac{B}{e^{-j\omega}-4}$$

$$= \frac{-8}{e^{-j\omega}-2} + \frac{8}{e^{-j\omega}-4} = \frac{(\frac{1}{2})\,8}{(\frac{1}{2})\,2-e^{-j\omega}} - \frac{(\frac{1}{4})\,8}{(\frac{1}{4})\,4-e^{-j\omega}} = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\xrightarrow{\mathcal{F}^{-1}} \boxed{4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]}$$

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer.)

$$Y(\omega) = H(\omega)X(\omega) \qquad \mathcal{F}(x[n]) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{= \frac{1}{1-\frac{1}{4}e^{-j\omega}}\left(\frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}\right)}$$

(10 pts) d) What is the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer)

$$\frac{1}{1-\frac{1}{4}e^{-j\omega}}\left(\frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}\right) = \frac{4}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

$$= \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

$$= \frac{-4}{1-\frac{1}{4}e^{-j\omega}} + \frac{8}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

$$\xrightarrow{\mathcal{F}^{-1}} \boxed{-4\left(\frac{1}{4}\right)^n u[n] + 8\left(\frac{1}{2}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n]}$$

(15 pts) **5.** Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2} dt.$$

by (20) by 5

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| X(\omega) \right|^2 d\omega \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \left[u(\omega+4) - u(\omega-4) \right] \pi \right|^2 d\omega$$

$$= \frac{\pi}{2}\int_{-4}^{4} d\omega = \frac{\pi}{2}\left(x \Big|_{-4}^{4} \right) = \frac{\pi}{2}(4+4) = 8\left(\frac{\pi}{2}\right) = \boxed{4\pi}$$