<u>Ex</u>: (Example 2.9 in text)

Assume we have an urn with 10 identical balls labeled $0, 1, \ldots, 9$. We pick a ball out of the urn and observe the number on the ball.

$S = \{0, 1, \ldots, 9\}$

$Pr(\{\text{picking ball } i\}) = 1/10, \quad i = 0, 1, \ldots, 9$

i) $A = \{1, 5, 9\}$ (picking ball 1 or ball 5 or ball 9)

$$
\begin{aligned}
Pr(A) &= Pr(\{1,5,9\}) \\
&= Pr(\{1\} \cup \{5\} \cup \{9\}) \\
&= Pr(\{1\}) + Pr(\{5\}) + Pr(\{9\}) \\
&= 3/10
\end{aligned}
$$

$\cancel{Pr(A)}$

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ii) $B = \{\text{ball number is even}\}$

$= \{0, 2, 4, 6, 8\}$

$$
\begin{aligned}
Pr(B) &= Pr(\{0,2,4,6,8\}) \\
&= Pr(\{0\} \cup \{2\} \cup \{4\} \cup \{6\} \cup \{8\}) \\
&= Pr(\{0\}) + Pr(\{2\}) + Pr(\{4\}) + \ldots + Pr(\{8\}) \\
&= 1/2
\end{aligned}
$$

iii) $C = \{\text{ball number is a multiple of } 3\}$

$= \{3, 6, 9\}$

$Pr(C) = 3/10$

iv) $D = A \cup C$ (A or C occur)

Note that $A \cap C = \{9\} \neq \emptyset$

therefore $Pr(D) \neq Pr(A) + Pr(C)$

Could use general formula for $Pr(A \cup C)$

$Pr(A \cup C) = Pr(A) + Pr(C) - Pr(A \cap C)$ (Check)

Instead, we can find D directly and compute $Pr(D)$

$D = \{1, 5, 9\} \cup \{3, 6, 9\}$

$= \{1, 3, 5, 6, 9\}$

$Pr(D) = 5/10 = 1/2$

Note: Decomposing events into unions of outcomes or mutually exclusive events allows us to easily compute probabilities using axiom (iii)

Ex: (2.12 from text)

Consider the random experiment
"pick a random number x between 0 and 1"

$S = [0,1]$ (continuous sample space)

$\Pr([a,b]) = b - a$ for $0 \le a \le b \le 1$

$[a,b]$: event that $x \in [a,b]$

i) $\Pr(\{x \text{ is between } 1/4 \text{ and } 1/3\})$

$= \Pr([1/4, 1/3])$

$= 1/3 - 1/4 = 1/12$

ii) $\Pr(\{x \text{ is at least } 1/4 \text{ away from } 1/2\})$

$= \Pr([0, 1/4] \cup [3/4, 1])$

$= \Pr([0, 1/4]) + \Pr([3/4, 1])$

$= 1/2$

iii) $\Pr((1/4, 1/3])$

$[1/4, 1/4]$

$[1/4, 1/3] = (1/4, 1/3] \cup \{1/4\}$

$\Rightarrow \Pr((1/4, 1/3]) = \Pr([1/4, 1/3]) - \Pr(\{1/4\})$

$= 1/12 - 0 = 1/2$

Note With the probability assignment above

$\Pr(\{x_0\}) = 0$ for $x_0 \in [0,1]$

Therefore

$\Pr([1/4, 1/3]) = \Pr((1/4, 1/3]) = \Pr([1/4, 1/3)) = \Pr((1/4, 1/3))$

Conditional Probability

~~Oftem~~ Often we are interested in determining whether knowledge of the occurrence of an event B affects the likelihood of the occurrence of an event A.

Ex roll a die and observe the face value

$S = \{1, 2, 3, 4, 5, 6\}$

$Pr(\{i\}) = 1/6, \quad i = 1, \dots, 6$

Assume we know the face value is even. Intuitively, the likelihood that the face value is ~~1~~ 2 is no longer $1/6$, and moreover it is greater than the likelihood that the face value 1. Can consider this as a "new" random experiment with a reduced sample space

$S' = \{2, 4, 6\}$

The outcomes of S' are equally likely so the probability that $i = 2$ should be $1/3$ if we know i is even

Let A, B be events with $Pr(B) > 0$.
The conditional probability of A given B is

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Properties of Conditional Probability

Let C be a fixed event with $Pr(C) > 0$.
Then we have that $Pr(\cdot \mid C)$ obeys the axioms of probability.

(i) $Pr(A \mid C) \geq 0$
(ii) $Pr(S \mid C) = 1$
(iii) If $A \cap B = \emptyset$ then
$$Pr(A \cup B \mid C) = Pr(A \mid C) + Pr(B \mid C)$$

Ex Assume we roll a fair die and denote the face value i as the outcome

$S = \{1, \dots, 6\}$, $Pr(\{i\}) = 1/6$, $i = 1, \dots, 6$,
$A = \{2\}$, $B = \{1\}$, $C = \{i \text{ is even}\}$

$$Pr(C) = Pr(\{2, 4, 6\}) = 1/2$$

$$Pr(A|C) = \frac{Pr(A \cap C)}{Pr(C)}$$

$\boxed{Pr(A|C) \neq Pr(C|A) \text{ in general}}$

$$= \frac{Pr(\{2\} \cap \{2,4,6\})}{1/2}$$

$$= Pr(\{2\}) / {}^1/_2$$

$$= {}^1/_6 / {}^1/_2 = {}^1/_3$$

$$Pr(B|C) = \frac{Pr(\{1\} \cap \{2,4,6\})}{Pr(C)}$$

$$= Pr(\emptyset)/Pr(C) = 0$$

Proposition: $Pr(\cdot|C)$ satisfies the axioms of probability

Proof: (i) $Pr(A|C) = \dfrac{Pr(A \cap C)}{Pr(C)}$

$Pr(A \cap C) \geq 0$ by axiom i), $Pr(C) > 0$

$\Rightarrow Pr(A|C) \geq 0$

ii)
$$Pr(S|C) = \frac{Pr(S\cap C)}{Pr(C)}$$
$$= \frac{Pr(C)}{Pr(C)}$$
$$= 1$$

iii)
$$Pr(A\cup B|C) = \frac{Pr((A\cup B)\cap C)}{Pr(C)}$$
$$= \frac{Pr((A\cap C)\cup(B\cap C))}{Pr(C)}$$

But $(A\cap C)\cap(B\cap C) = (A\cap B)\cap C = \phi\cap C = \phi$

$$\Rightarrow Pr(A\cup B|C) = \frac{Pr(A\cap C)}{Pr(C)} + \frac{Pr(B\cap C)}{Pr(C)}$$
$$= Pr(A|C) + Pr(B|C)$$

Bayes Rule

Let A and B be events in the space S with $Pr(A), Pr(B) > 0$

From the definition of conditional probability we can write $Pr(A \cap B)$ in the following ways:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

$$\Rightarrow \quad Pr(A \cap B) = Pr(A|B)\, Pr(B) \qquad (1)$$

$$Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)}$$

$$\Rightarrow \quad Pr(A \cap B) = Pr(B|A)\, Pr(A) \qquad (2)$$

Combining (1) and (2) through transitivity, we get

$$Pr(A|B)\, Pr(B) = Pr(B|A)\, Pr(A)$$

$$\Rightarrow \quad \boxed{Pr(A|B) = \frac{Pr(B|A)\, Pr(A)}{Pr(B)}} \qquad \text{Bayes Rule}$$

Bayes rule allows us to find $P(A|B)$ in terms of $Pr(B|A)$, $Pr(A)$, and $Pr(B)$