1. O+W 9.21 (b, d, h) Find LTs, RoCs, and pole-zero plots.

(b) $x(t) = e^{-4t}u(t) + e^{-5t}\sin 5t\, u(t)$.

This can be obtained from Table 9.2

$$e^{-4t}u(t) \longleftrightarrow \frac{1}{s+4} \qquad Re(s) > -4$$

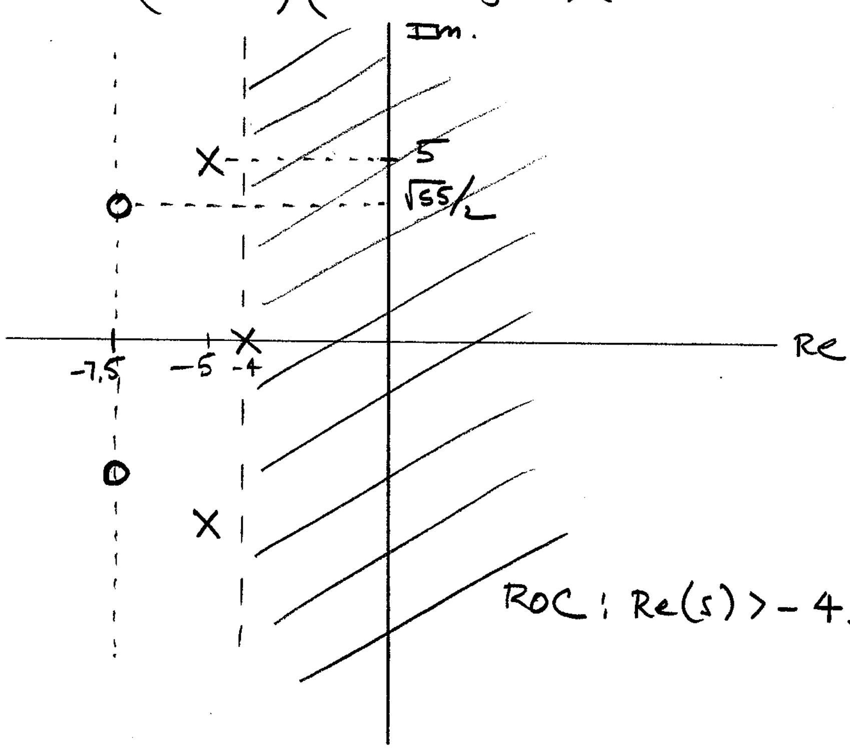
$$e^{-5t}\sin 5t\, u(t) \longleftrightarrow \frac{5}{(s+5)^2+25} \qquad Re(s) > -5$$

$$\therefore X(s) = \frac{1}{s+4} + \frac{5}{(s+5)^2+25} \qquad Re(s) > -4$$

(intersect. of the two RoCs).

$$= \frac{s^2+15s+70}{(s+4)(s^2+10s+50)}$$

$$= \frac{\left(s+\frac{15}{2}+j\frac{\sqrt{55}}{2}\right)\left(s+\frac{15}{2}-j\frac{\sqrt{55}}{2}\right)}{(s+4)(s+5+j5)(s+5-j5)}.$$



RoC: $Re(s) > -4$.

(d) $x(t) = t e^{-2|t|}$

$$= \begin{cases} t e^{-2t} & t > 0 \\ t e^{2t} & t < 0. \end{cases}$$

$$= t e^{-2t} u(t) + t e^{2t} u(-t).$$

Both of these may be found in Table 9.2.

$$t e^{-2t} u(t) \leftrightarrow \frac{1}{(s+2)^2} \quad \mathrm{Re}(s) > -2$$

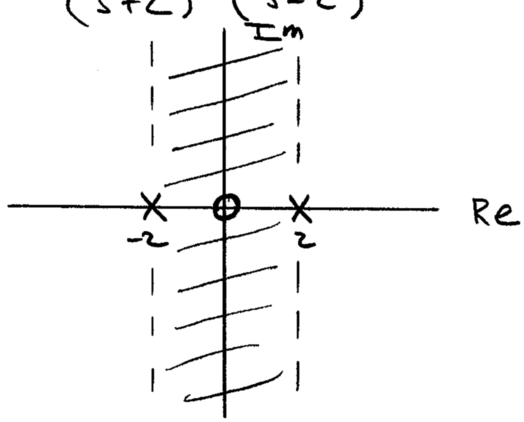
$$t e^{2t} u(-t) \leftrightarrow -\frac{1}{(s-2)^2} \quad \mathrm{Re}(s) < 2$$

$$\therefore \; X(s) = \frac{1}{(s+2)^2} - \frac{1}{(s-2)^2} \quad -2 < \mathrm{Re}(s) < 2$$

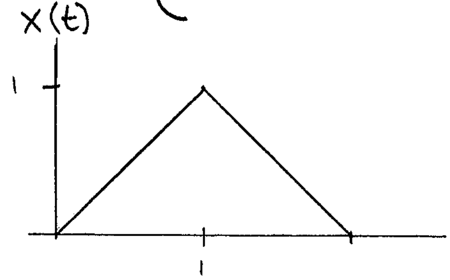
$$= \frac{(s-2)^2 - (s+2)^2}{(s+2)^2(s-2)^2}$$

$$= \frac{s^2 - 4s + 4 - s^2 - 4s - 4}{(s+2)^2(s-2)^2}$$

$$= \frac{-8s}{(s+2)^2(s-2)^2}$$



(h) $x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$



We could directly evaluate the LT. It would require an integration by parts and some simplification

Another approach is to put the transform together in pieces using various transform properties. Since this latter method gives experience using the transform properties, we will use it. Let

$$x(t) = \underbrace{x_1(t)}_{= \begin{cases} t & 0 \le t \le 1 \\ 0 & \text{else} \end{cases}} + \underbrace{x_2(t)}_{= \begin{cases} 2-t & 1 \le t \le 2 \\ 0 & \text{else.} \end{cases}}$$

Let $x_3(t) = x_1(t+2)$ and then note that $x_2(t) = x_3(-t)$. (so can start with the transform of $x_1(t)$ and get the rest using various properties).

Note $x_1(t) = t \cdot \{u(t) - u(t-1)\} \Rightarrow$ will use the $\frac{d}{ds}$ property!

$$u(t) - u(t-1) \longleftrightarrow \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1-e^{-s}}{s}$$

Using $\frac{d}{ds}$ property

$$t \cdot \{u(t) - u(t-1)\} \longleftrightarrow -\frac{d}{ds}\left\{\frac{1-e^{-s}}{s}\right\}$$

$$= \frac{1-(s+1)e^{-s}}{s^2} \qquad \text{RoC} = \mathbb{C}$$

$$= X_1(s).$$

Now $x_3(t) = x_1(t+2)$. Therefore

$$X_3(s) = e^{2s} X_1(s) = \frac{e^{2s}}{s^2}\left[1-(s+1)e^{-s}\right] \quad \text{ROC} = \mathbb{C}.$$

Finally $x_2(t) = x_3(-t)$ so $X_2(s) = X_3(-s)$.

$$\Longrightarrow \quad X_2(s) = \frac{e^{-2s}}{s^2}\left[1-(1-s)e^{s}\right] \quad \text{ROC} = \mathbb{C}.$$

Summing the two pieces

$$x(t) = x_1(t) + x_2(t)$$

$$\updownarrow$$

$$X(s) = \frac{1-(s+1)e^{-s}}{s^2} + \frac{e^{-2s}\left[1-(1-s)e^{s}\right]}{s^2}$$

$$= \frac{\left(1-e^{-s}\right)^2}{s^2} \qquad \text{ROC} = \mathbb{C}.$$

This function has a double zero and a double pole at $s=0$ $\Longrightarrow$ They cancel ... said to be a <u>removable singularity</u>.

2. O+W 9.22 (b,d) Find time function given LT and RoC.

(b) $\frac{s}{s^2+9}$ $Re(s) < 0$.

$$\parallel$$

$$\frac{s}{(s+j3)(s-j3)} = \frac{A}{s+j3} + \frac{B}{s-j3}$$

$$A = \frac{-j3}{-j3-j3} = \frac{1}{2} = B.$$

Therefore $X(s) = \frac{1/2}{s+j3} + \frac{1/2}{s-j3}$ $Re(s) < 0$ for both.

These are anticausal RoCs so substitute $s \mapsto -s$ to get

$$\tilde{X}(s) = \frac{1/2}{-s+j3} + \frac{1/2}{-s-j3} = \frac{-1/2}{s-j3} + \frac{-1/2}{s+j3} \quad Re(s)>0$$

$$\updownarrow$$

$$-\frac{1}{2}e^{+j3t}u(t) - \frac{1}{2}e^{-j3t}u(t).$$

Then substitute $t \mapsto -t$ to get

$$x(t) = \left(-\frac{1}{2}e^{-j3t} - \frac{1}{2}e^{+j3t}\right)u(-t).$$

$$= -\cos 3t \; u(-t).$$

(d) $\frac{s+2}{s^2+7s+12}$ $-4 < Re(s) < -3$

$$\parallel$$

$$X(s) = \frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A = \left.\frac{s+2}{s+4}\right|_{s=-3} = \frac{-1}{1} = -1$$

$$B = \left.\frac{s+2}{s+3}\right|_{s=-4} = \frac{-2}{-1} = 2$$

$$X(s) = \frac{-1}{s+3} + \frac{2}{s+4}$$

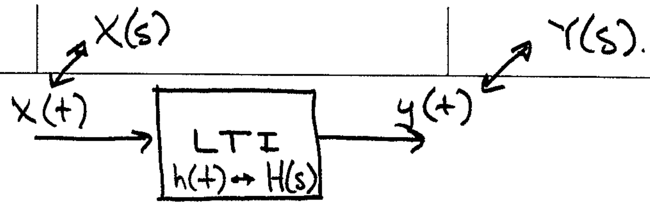
$Re(s) > -4$ ie causal.

$Re(s) < -3$ ie anti-causal.

From Table 9.2

$$x(t) = e^{-3t}u(-t) + 2e^{-4t}u(t).$$

3. O+W 9.31



$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y(t) = x(t)$$

(a) Find H(s) and sketch its pole-zero plot.

Take LT of both sides of the differential equation

$$s^2 Y(s) - sY(s) - 2Y(s) = X(s)$$

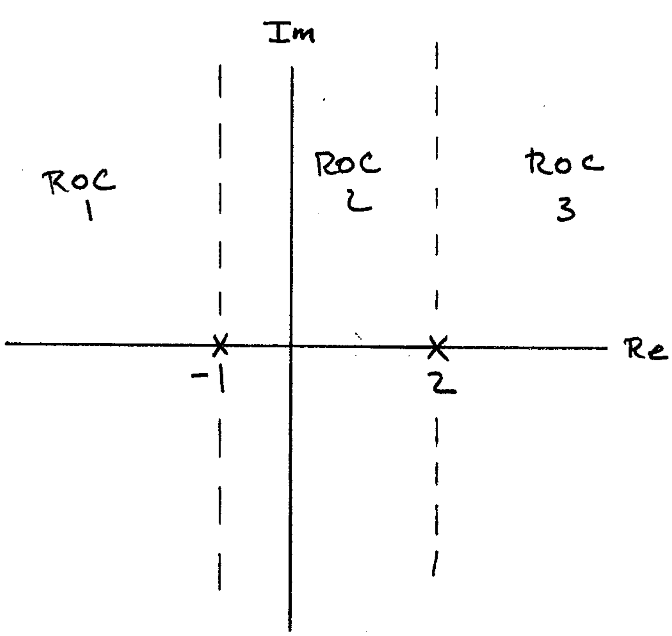
$$\Longrightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s+1)(s-2)}$$

$$= \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = \left.\frac{1}{s-2}\right|_{s=-1} = -\frac{1}{3}$$

$$B = \left.\frac{1}{s+1}\right|_{s=2} = \frac{1}{3}$$

$$\Longrightarrow H(s) = \frac{-1/3}{s+1} + \frac{1/3}{s-2}$$



There are 3 possible ROCs.

(b) Determine $h(t)$ for three cases.

(1) Stable system $\Longrightarrow$ must use ROC 2 $-1 < Re(s) < 2$

$\frac{1}{s+1}$ $Re(s) > -1$ $\longleftrightarrow$ $e^{-t}u(t)$
ie causal

$\frac{1}{s-2}$ $Re(s) < 2$ $\longleftrightarrow$ $-e^{2t}u(-t)$.
ie anticausal

Sum and scale:

$$h(t) = -\frac{1}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t).$$

(2) causal syst. $\Longrightarrow$ must use ROC 3 $Re(s) > 2$

$\frac{1}{s+1}$ term same as in (1).

$\frac{1}{s-2}$ $Re(s) > 2$ $\longleftrightarrow$ $e^{2t}u(t)$.
ie causal

Sum and scale:

$$h(t) = -\frac{1}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t).$$

(3) neither stable nor causal $\Longrightarrow$ must use ROC 1, $Re(s) < -1$

$\frac{1}{s-2}$ term same as (1).

$\frac{1}{s+1}$ $Re(s) < -1$ $\longleftrightarrow$ $-e^{-t}u(-t)$

Sum and scale:

$$h(t) = \frac{1}{3}e^{-t}u(-t) - \frac{1}{3}e^{2t}u(-t).$$

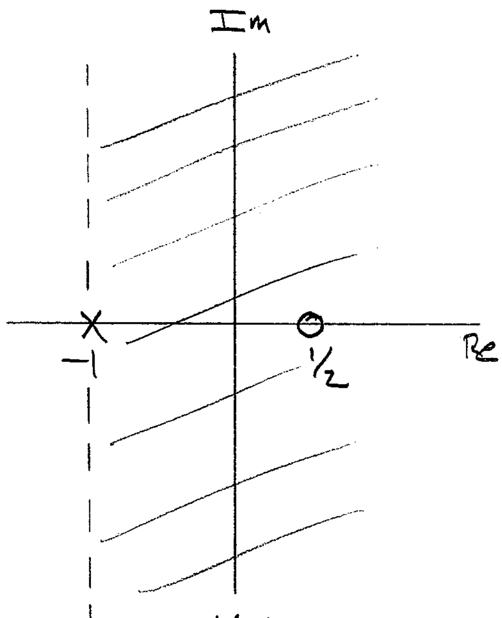
4. O+W 9.48

(a) The algebraic relationship between a system $H(s)$ and its inverse system $H_1(s)$ is

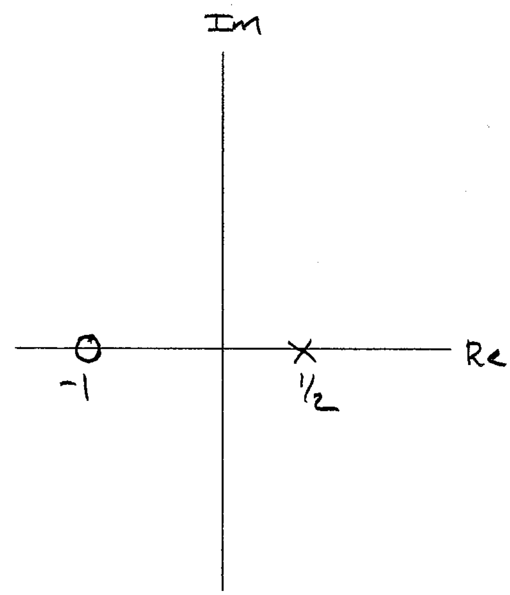
$$H_1(s) = 1/H(s).$$

(b) The pole-zero plot for $H(s)$ is as shown below. We are also told that its ROC corresponds to a stable and causal system.

What must be the pole-zero plot of $H_1(s)$ and its associated ROC?



$H(s)$



$H_1(s)$.

$$H(s) = K\,\frac{s-1/2}{s+1} \quad Re(s) > -1.$$

$\Longrightarrow H_1(s) = K^{-1}\,\dfrac{s+1}{s-1/2}$. In order that the cascade $H(s)H_1(s) = 1$ correspond to a time-domain convol. equation that is absolutely summable, the ROCs must intersect. Here both ROCs for $H_1(s)$ will intersect:

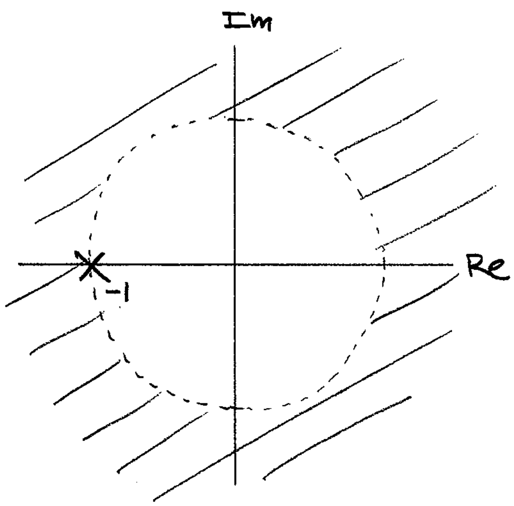
$Re(s) > 1/2 \Longrightarrow h_1(t)$ is causal unstable, $Re(s) < 1/2 \Longrightarrow h_1(t)$ anticausal stable.

5. <u>O+W 10.21 (c,e)</u> Find Z-transform, pole-zero plot, and ROC.

(c) $x[n] = (-1)^n u[n]$

From Table 10.2 (or direct computation)

$$X(z) = \frac{1}{1+z^{-1}} \qquad |z| > 1$$



Re: DTFT unit circle is boundary of ROC. Thus DTFT must include a delta function. Can show

$$X(e^{j\omega}) = \frac{1}{1+e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi - 2\pi k).$$

(e) $x[n] = \left(-\frac{1}{3}\right)^n u[-n-2]$

Could use the Table but would need to shift or drop a term. Easier to compute directly:

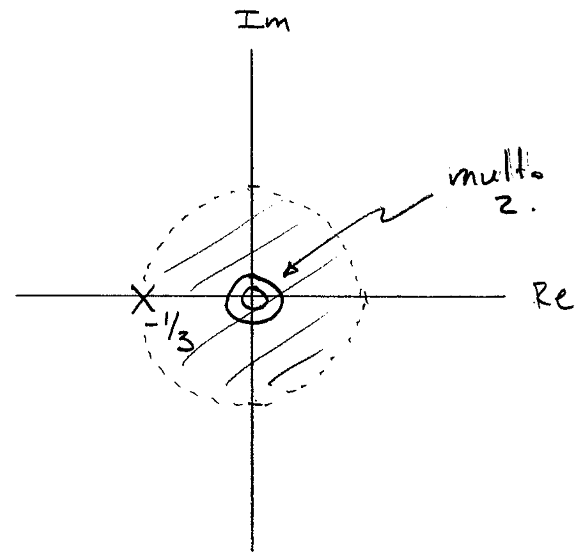
$$X(z) = \sum_{n=-\infty}^{-2} \left(-\tfrac{1}{3}\right)^n z^{-n} \underset{(k=-n)}{=} \sum_{k=2}^{\infty} \left(-\tfrac{1}{3}\right)^{-k} z^k$$

$$\underset{(\ell=k-2)}{=} \sum_{\ell=0}^{\infty} \left(-\tfrac{1}{3}\right)^{-(\ell+2)} z^{\ell+2} = \left(-\tfrac{1}{3}\right)^{-2} z^2 \sum_{\ell=0}^{\infty} \left[\left(-\tfrac{1}{3}\right)^{-1} z\right]^{\ell}$$

$$= 9z^2 \sum_{\ell=0}^{\infty} (-3z)^{\ell} = 9z^2 \frac{1}{1+3z} \qquad |-3z| < 1$$

$$= \frac{9z^2}{3(z+1/3)} = \frac{3z^2}{z+\frac{1}{3}} = \frac{3z}{1+\frac{1}{3}z^{-1}} = \frac{3}{z^{-1}(1+\frac{1}{3}z^{-1})}$$

provided $|z| < 1/3$.



Re: DTFT ROC does not include unit circle nor is unit circle a boundary of ROC.

Therefore, no DTFT

6. O+W 10.22 (b) Same as prev.

(b) $x[n] = n\left(\frac{1}{2}\right)^{|n|}$.

First consider $\tilde{x}[n] = \left(\frac{1}{2}\right)^{|n|} = \left(\frac{1}{2}\right)^{n} u[n] + \left(\frac{1}{2}\right)^{-n} u[-n-1]$

From Table 10.2

$\frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > 1/2$

$\frac{-1}{1-2z^{-1}} \quad |z| < 2$

The ROCs of the parts overlap. Therefore

$$\tilde{X}(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \qquad 1/2 < |z| < 2$$

$$= \frac{z}{z-1/2} - \frac{z}{z-2}$$

Since $x[n] = n\tilde{x}[n]$ can use the diff. in z-domain property to conclude

$$X(z) = -z\,\frac{d\tilde{X}(z)}{dz}.$$

$$\frac{d}{dz}\left(\frac{z}{z-1/2}\right) = \frac{(z-1/2)-z}{(z-1/2)^2} = \frac{-1/2}{(z-1/2)^2}$$

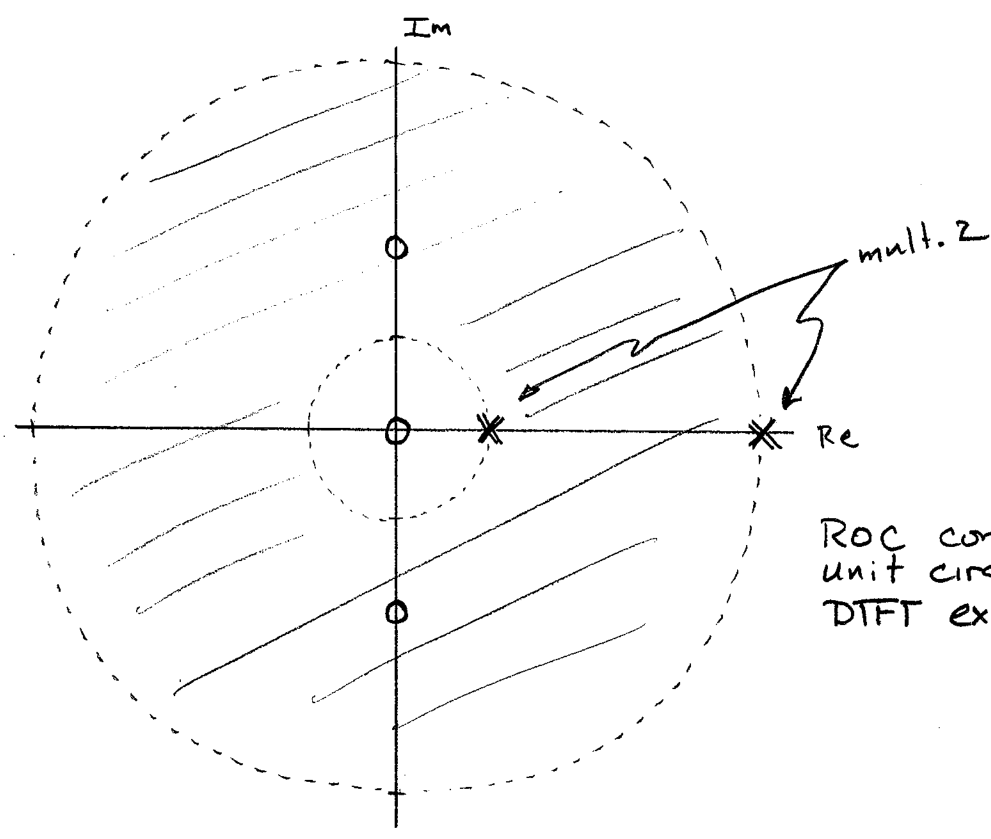
$$\frac{d}{dz}\left(\frac{z}{z-2}\right) = \frac{(z-2)-z}{(z-2)^2} = \frac{-2}{(z-2)^2}$$

$$\therefore \quad X(z) = \frac{\frac{1}{2}z}{(z-1/2)^2} - \frac{2z}{(z-2)^2} \qquad \tfrac{1}{2} < |z| < 2$$

$$= \frac{\frac{1}{2}z\,(z-2)^2 - 2z\,(z-\frac{1}{2})^2}{(z-1/2)^2\,(z-2)^2}$$

$$= \frac{-\frac{3}{2}z\,(z^2+1)}{(z-1/2)^2\,(z-2)^2} = \frac{-\frac{3}{2}z\,(z+j)(z-j)}{(z-\frac{1}{2})^2\,(z-2)^2}$$

$$= \frac{-\frac{3}{2}z^{-1}\,(1+z^{-2})}{(1-\frac{1}{2}z^{-1})^2\,(1-2z^{-1})^2} \qquad \tfrac{1}{2} < |z| < 2$$



ROC contains unit circle so DTFT exists

7. O + W 10.24 Using various methods, find inverse Z transform.

(a) use partial fractions

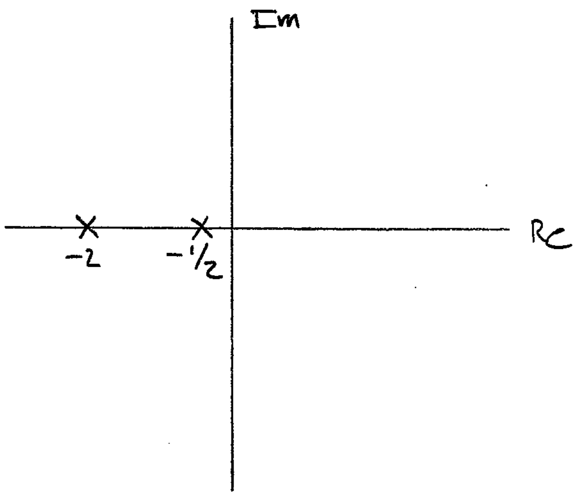
$$X(z) = \frac{1 - 2z^{-1}}{1 + \frac{5}{2}z^{-1} + z^{-2}} \qquad x[n] \text{ abs. summable.}$$

$$= \frac{1-2z^{-1}}{(1+\frac{1}{2}z^{-1})(1+2z^{-1})} = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1+2z^{-1}}$$

$$A = \left.\frac{1-2z^{-1}}{1+2z^{-1}}\right|_{z=-1/2} = \frac{1-2(-2)}{1+2(-2)} = \frac{5}{-3} = -\frac{5}{3}$$

$$B = \left.\frac{1-2z^{-1}}{1+\frac{1}{2}z^{-1}}\right|_{z^{-1}=-\frac{1}{2}} = \frac{1-2(-1/2)}{1+\frac{1}{2}(-1/2)} = \frac{2}{3/4} = \frac{8}{3}$$

$$X(z) = \frac{-5/3}{1+\frac{1}{2}z^{-1}} + \frac{8/3}{1+2z^{-1}}$$



In order that $x[n]$ be absolutely summable must have

$$\tfrac{1}{2} < |z| < 2$$

$$\frac{-5/3}{1+\frac{1}{2}z^{-1}} \quad |z| > 1/2 \longleftrightarrow -\frac{5}{3}\left(-\frac{1}{2}\right)^n u[n]$$

$$\frac{8/3}{1+2z^{-1}} \quad |z| < 2 \longleftrightarrow -\frac{8}{3}(-2)^n u[-n-1].$$

$$\therefore\ x[n] = -\frac{5}{3}\left(-\frac{1}{2}\right)^n u[n] - \frac{8}{3}(-2)^n u[-n-1].$$

(b) use long division

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}} \quad \text{and } x[n] \text{ is right-sided.}$$

$$
\begin{array}{r l}
 & 1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} \ldots \\
1+\frac{1}{2}z^{-1} \,\big) & 1 - \frac{1}{2}z^{-1} \\
 & \underline{1 + \frac{1}{2}z^{-1}} \\
 & -z^{-1} \\
 & \underline{-z^{-1} - \frac{1}{2}z^{-2}} \\
 & \frac{1}{2}z^{-2} \\
 & \underline{\frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}} \\
 & -\frac{1}{4}z^{-3} \\
 & \underline{-\frac{1}{4}z^{-3} - \frac{1}{8}z^{-4}} \\
 & \frac{1}{8}z^{-4}
\end{array}
$$

$$\therefore \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}} = 1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} \ldots$$

$$= 1 + \sum_{n=0}^{\infty} (-1)\left(-\frac{1}{2}\right)^{n} z^{-n+1}$$

Therefore the time domain signal can be written

$$x[n] = \delta[n] - \left(-\frac{1}{2}\right)^{n-1} u[n-1]$$

(c) Use partial fractions

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}} \quad \text{and } x[n] \text{ is abs. summable}$$

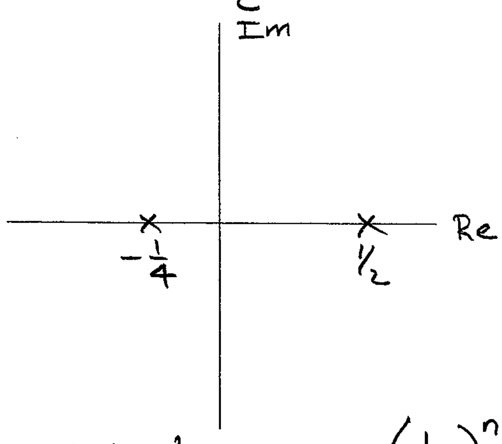
$$= \frac{3z}{z^2 - \frac{1}{4}z - \frac{1}{8}} = \frac{3z}{(z-\frac{1}{2})(z+\frac{1}{4})}$$

$$= \frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{4}z^{-1}}$$

$$A = \left.\frac{3z^{-1}}{1+\frac{1}{4}z^{-1}}\right|_{z^{-1}=2} = \frac{6}{1+\frac{1}{2}} = \frac{6}{3/2} = \frac{12}{3} = 4$$

$$B = \left.\frac{3z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=-4} = \frac{-12}{1-\frac{1}{2}(-4)} = \frac{-12}{3} = -4$$

$$X(z) = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{4}{1+\frac{1}{4}z^{-1}}$$



To be abs. summable the ROC must include unit circle.

So each term must have causal (outside disk) ROC.

$$\frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \longleftrightarrow \left(\frac{1}{2}\right)^n u[n]$$

$$\frac{1}{1+\frac{1}{4}z^{-1}} \quad |z| > \frac{1}{4} \longleftrightarrow \left(-\frac{1}{4}\right)^n u[n].$$

$$\therefore \quad x[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{4}\right)^n u[n].$$

8. O+W 10.34 Causal LTI system has difference equation.

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

(a) Find system function, pole-zero plot, and ROC.

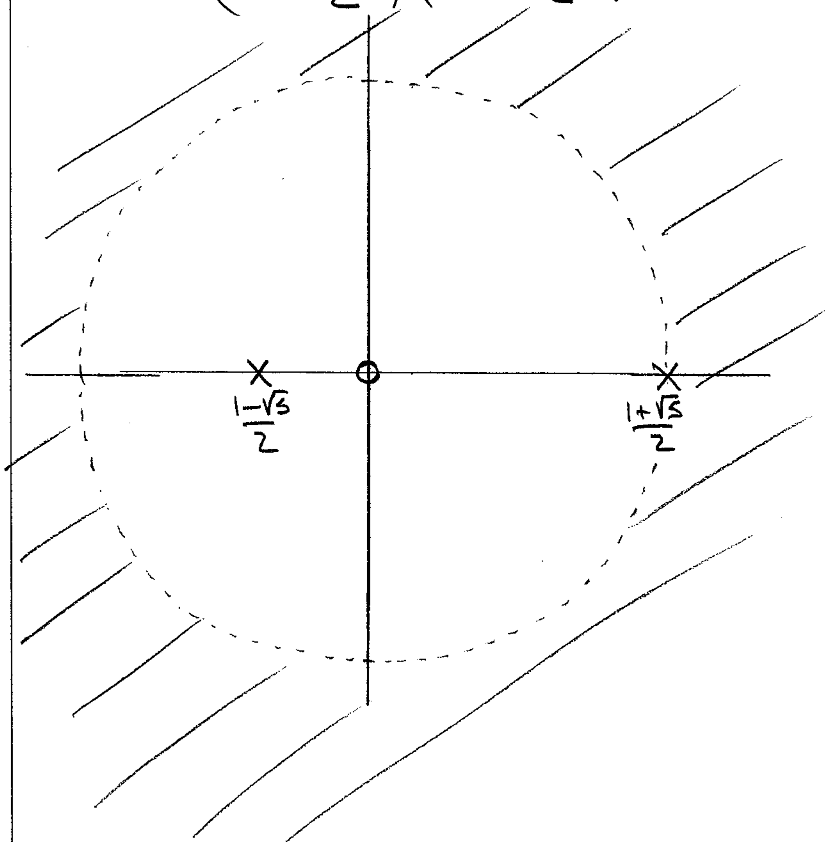
$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1}$$

$$= \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}$$

$\frac{1+\sqrt{5}}{2} \approx 1.62$

$\frac{1-\sqrt{5}}{2} \approx -0.62$



ROC must be $|z| > \frac{1+\sqrt{5}}{2}$ since causal.

(b) Find $h[n]$.

$$H(z) = \frac{z^{-1}}{\left(1 - \left(\frac{1+\sqrt{5}}{2}\right)z^{-1}\right)\left(1 - \left(\frac{1-\sqrt{5}}{2}\right)z^{-1}\right)} = \frac{A}{1 - \left(\frac{1+\sqrt{5}}{2}\right)z^{-1}} + \frac{B}{1 - \left(\frac{1-\sqrt{5}}{2}\right)z^{-1}}$$

$$A = \left.\frac{z^{-1}}{1-\left(\frac{1-\sqrt{5}}{2}\right)z^{-1}}\right|_{z^{-1}=\frac{2}{1+\sqrt{5}}} = \frac{\frac{2}{1+\sqrt{5}}}{1-\frac{1-\sqrt{5}}{1+\sqrt{5}}}$$

$$= \frac{2}{1+\sqrt{5}-1+\sqrt{5}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$B = \left.\frac{z^{-1}}{1-\left(\frac{1+\sqrt{5}}{2}\right)z^{-1}}\right|_{z^{-1}=\frac{2}{1-\sqrt{5}}} = \frac{\frac{2}{1-\sqrt{5}}}{1-\frac{1+\sqrt{5}}{1-\sqrt{5}}}$$

$$= \frac{2}{1-\sqrt{5}-1-\sqrt{5}} = \frac{2}{-2\sqrt{5}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

$$\therefore H(z) = \frac{\sqrt{5}/5}{1-\left(\frac{1+\sqrt{5}}{2}\right)z^{-1}} - \frac{\sqrt{5}/5}{1-\left(\frac{1-\sqrt{5}}{2}\right)z^{-1}}$$

$$|z| > \frac{1+\sqrt{5}}{2} \qquad\qquad |z| > \left|\frac{1-\sqrt{5}}{2}\right|$$

$$\updownarrow$$

$$h[n] = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

(c) Above impulse response is unstable. To find a stable impulse response must pick ROC to include unit circle ie

$$\frac{\sqrt{5}-1}{2} < |z| < \frac{\sqrt{5}+1}{2}$$

This will only change the term involving the pole at

$$z = \frac{1+\sqrt{5}}{2}$$

$$\frac{\sqrt{5}/5}{1-\left(\frac{1+\sqrt{5}}{2}\right)z^{-1}} \qquad |z| < \frac{1+\sqrt{5}}{2}$$

$$\Updownarrow$$

$$-\frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n} u[-n-1]$$

$$\therefore\ h_{stable}[n] = -\frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n} u[-n-1] - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n} u[n].$$