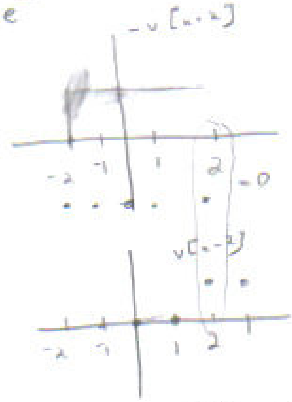
(15 pts) 1. Compute the Fourier transform of the DT signal

$$x[n] = n^2u[n-2] - n^2u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$X(\omega) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\sum_{k=-\infty}^{\infty} n^2 \left(u[n-2] - u[n+2] \right) e^{-j\omega n}$$



$$= \sum_{n=-2}^{1} -n^2 e^{-j\omega n}$$

$$= -4e^{2j\omega} - (e^{j\omega} + e^{-j\omega} = 2\cos(\omega))$$

$$\boxed{X(\omega) = -4e^{2j\omega} - 2\cos(\omega)}$$

$= -1$ from nn

$-2 \le n \le 1$

(15 pts) **2.** Show that the Fourier transform of the CT signal $x(t) = \cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

Show that

$$\mathcal{F}^{-1}(X(\omega)) = \cos(\omega_0 t)$$

$$\mathcal{F}^{-1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\cancel{\pi}}\int_{-\infty}^{\infty} \cancel{\pi}(\delta(\omega+\omega_0) + \delta(\omega-\omega_0))e^{j\omega t}$$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \quad \text{by "sifting property"}$$

$$= \cos(\omega_0 t)$$

$g[n]^2 = j\,x(n)$

$= -x(n)$

(15 pts) **3.** Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{\cos\omega}$. Explain why Bob's answer is wrong.

x[n] is real and even

$\Rightarrow$ X(ω) must be real and even (35)

so Bob is wrong since his answer is imag.

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\sin\omega}$. Could Alice be right? Explain.

x[n] is real and even

$\Rightarrow$ X(ω) must be real and even (35)

but Alice's X(ω) is real and odd

Alice is wrong

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

All DT FT are periodic with period 2π.

His answer is not periodic

Devin is wrong

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

$$\Rightarrow \mathcal{F}^{-1}\left(Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega)\right) = \mathcal{F}^{-1}\left(2X(\omega)\right)$$

$$= y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

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(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

$$Y(\omega) = X(\omega) H(\omega)$$

$$= \frac{1}{\left(1 - \frac{1}{4}e^{-j\omega n}\right)} \cdot \frac{2}{\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)}$$

$$= \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega n}\right)^2 \left(1 - \frac{1}{2}e^{-j\omega}\right)}$$

(15 pts) b) Find the unit impulse response of this system.

$$h[n] = \mathcal{F}^{-1}\left(H(\omega)\right)$$

$$= \mathcal{F}^{-1}\left(\frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}\right)$$

$$= \mathcal{F}^{-1}\left(\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}\right)$$

$$= A\left(\frac{1}{4}\right)^n u[n] + B\left(\frac{1}{2}\right)^n u[n]$$

$$A = -2 \quad B = 4$$

$$\boxed{h[n] = -2\left(\frac{1}{4}\right)^n u[n] + 4\left(\frac{1}{2}\right)^n u[n]}$$

(20 pts) **5.** Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

$$x(at+b) \xrightarrow{\mathcal{F}} \frac{e^{j\omega\frac{b}{a}}}{-a} X\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$\mathcal{F}\left(x(at+b)\right) = \int_{-\infty}^{\infty} x(at+b)\, e^{-j\omega t}\, dt$$

usually solve for $x(t)$

so let $r = at + b \qquad t = \frac{r}{a} - \frac{b}{r}$

$$\frac{dr}{dt} = a \qquad dt = \frac{dr}{a}$$

$$\frac{1}{a} \int_{-\infty}^{\infty} x(r)\, e^{j\omega\left(\frac{r}{a} - \frac{b}{a}\right)}\, dr$$

$$= \frac{1}{a} e^{-\frac{j\omega b}{a}} \int_{\infty}^{-\infty} x(r)\, e^{\frac{j\omega r}{a}}\, dr$$

choose $\omega' = \frac{\omega}{a}$

$$\int_{\infty}^{-\infty} x(r) e^{j\omega' r}\, dr$$

$= 1 \; e^{-\frac{j\omega b}{a}}$ [illegible] $\qquad 1\; e^{-\frac{j\omega b}{a}} X\left(\frac{\omega}{a}\right)$