ECE 301
Fall 2008
Instructor: Mimi Boutin
Midterm Examination 3

Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.
2. You have 50 minutes to answer the 5 questions contained in this exam. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.
3. This booklet contains 12 pages. The last five pages contain a table of formulas and properties. You may tear out these pages **once the exam begins**. Each transform and each property is labeled with a number. To save time, you may use these numbers to specify which transform/property you are using when justifying your answer. In general, if you use a non-trivial fact that is *not* contained in this table, you must explain why it is true in order to get full credit. The only exceptions are the properties of the ROC, which can be used without justification.
4. This is a closed book exam. The only personal items allowed are pens/pencils, erasers and something to drink. Anything else is strictly forbidden.

Itemized Scores
Problem 1:
Problem 2:
Problem 3:
Problem 4:
Problem 5:
Total:

Name:________________________________

Email:________________________________

Signature:____________________________

1. (10 pts) a) Use the definition of the Laplace transform (Eq. 39) to compute the Laplace transform (with ROC) of

$$x(t) = e^{(-2+3j)t}u(t).$$

(5 pts) b) Use your answer in a) to obtain the Fourier transform of

$$x(t) = e^{(-2+3j)t}u(t).$$

2. (10 pts) a) Use the definition of the z-transform (Eq. 54) to compute the z-transform (with ROC) of

$$x[n] = (1+2j)^n \left(u[n] - u[n-2]\right).$$

(5 pts) b) Use your answer in a) to obtain the Fourier transform of

$$x[n] = (1+2j)^n \left(u[n] - u[n-2]\right),$$

.

(25 pts) **3.** Consider the signal $x(t) = \frac{\sin(\pi t)}{t} e^{j2\pi t}$.
a) Is $x(t)$ band limited? If so, what is its nyquist rate? If not, explain why.

b) Define the discrete signal $x_1[n] = x(\frac{1}{10}n)$, for all integers n. Can one recover $x(t)$ from $x_1[n]$? Justify your answer.

c) Define the discrete signal $x_2[n] = x(\frac{2}{3}n)$, for all integers n. Can one recover $x(t)$ from $x_2[n]$? Justify your answer.

b) Define the discrete signal $x_3[n] = x(2n)$, for all integers n. Can one recover $x(t)$ from $x_3[n]$? Justify your answer.

(10 pts) **4.** A CT signal $x(t)$ with Nyquist rate equal to 6π is the input of a CT system with frequency response $H_c(j\omega)$ given by

$$H_c(j\omega) = \begin{cases} |\omega|, & \text{if } |\omega| < 3\pi, \\ 0, & else. \end{cases}$$

Mary would like to process $x(t)$ in an equivalent fashion by first sampling this signal, then processing the samples with a DT system, and reconstructing a CT signal from the processed samples. Can this work? If so, what should the frequency response of the DT system be? (Drawing a sketch is sufficient.) If not, explain why.

(10 pts) **5.** A CT signal $x(t)$ with Nyquist rate equal to 6π is the input of a CT system with frequency response $H_c(j\omega)$ given by

$$H_c(j\omega) = \begin{cases} j\omega, & \text{if } \omega > 0 \\ -j\omega, & else. \end{cases}$$

Joe would like to process $x(t)$ in an equivalent fashion by first sampling this signal, then processing the samples with a DT system, and reconstructing a CT from the processed samples. Can this work? If so, what should the frequency response of the DT system be? (Drawing a sketch is sufficient.) If not, explain why.

Table

Fourier Series of CT Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \quad (1)$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt \quad (2)$$

Some CT Fourier series

Signal	a_k	
$e^{j\omega_0 t}$	$a_1 = 1, a_k = 0$ else.	(3)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0 & T_1 < \lvert t\rvert < \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\frac{\sin k\omega_0 T_1}{k\pi}$	(4)
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$a_k = \frac{1}{T}$	(5)

Fourier Series of DT Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \quad (6)$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} \quad (7)$$

CT Fourier Transform

$$\text{F.T.}: \mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{8}$$

$$\text{Inverse F.T.}: x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{9}$$

Properties of CT Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

	Signal	F.T.	
Linearity:	$ax(t) + by(t)$	$a\mathcal{X}(\omega) + b\mathcal{Y}(\omega)$	(10)
Time Shifting:	$x(t - t_0)$	$e^{-j\omega t_0}\mathcal{X}(\omega)$	(11)
Frequency Shifting:	$e^{j\omega_0 t}x(t)$	$\mathcal{X}(\omega - \omega_0)$	(12)
Time and Frequency Scaling:	$x(at)$	$\frac{1}{\lvert a\rvert}\mathcal{X}\left(\frac{\omega}{a}\right)$	(13)
Multiplication:	$x(t)y(t)$	$\frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega)$	(14)
Convolution:	$x(t) * y(t)$	$\mathcal{X}(\omega)\mathcal{Y}(\omega)$	(15)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$j\omega\mathcal{X}(\omega)$	(16)

Some CT Fourier Transform Pairs

$$e^{j\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0) \tag{17}$$

$$1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega) \tag{18}$$

$$\frac{\sin Wt}{\pi t} \xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W) \tag{19}$$

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \tag{20}$$

$$u(t + T_1) - u(t - T_1) \xrightarrow{\mathcal{F}} \frac{2\sin(\omega T_1)}{\omega} \tag{21}$$

$$\sum_{n=-\infty}^{\infty}\delta(t - nT) \xrightarrow{\mathcal{F}} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{T}) \tag{22}$$

DT Fourier Transform

$$\text{F.T.:}\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (23)$$

$$\text{Inverse F.T.: } x[n] = \frac{1}{2\pi}\int_{2\pi} \mathcal{X}(\omega)e^{j\omega n}d\omega \quad (24)$$

Properties of DT Fourier Transform

Let $x(t)$ be a signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

	Signal	F.T.	
Linearity:	$ax[n] + by[n]$	$a\mathcal{X}(\omega) + b\mathcal{Y}(\omega)$	(25)
Time Shifting:	$x[n - n_0]$	$e^{-j\omega n_0}\mathcal{X}(\omega)$	(26)
Frequency Shifting:	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(\omega - \omega_0)$	(27)
Time Reversal:	$x[-n]$	$\mathcal{X}(-\omega)$	(28)
Time Exp.:	$x_k[n] = \begin{cases} x[\frac{n}{k}], & \text{if } k \text{ divides } n \\ 0, & \text{else.} \end{cases}$	$\mathcal{X}(\omega)$	(29)
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega)$	(30)
Convolution:	$x[n] * y[n]$	$\mathcal{X}(\omega)\mathcal{Y}(\omega)$	(31)
Differencing in Time:	$x[n] - x[n-1]$	$(1 - e^{-j\omega})\mathcal{X}(\omega)$	(32)

Some DT Fourier Transform Pairs

$$\sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \quad (33)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \quad (34)$$

(35)

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \quad (36)$$

$$\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{1 - \alpha e^{-j\omega}} \quad (37)$$

(38)

Laplace Transform

$$X(s) \quad = \quad \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{39}$$

Properties of Laplace Transform

Let $x(t)$, $x_1(t)$ and $x_2(t)$ be three CT signals and denote by $X(s)$, $X_1(s)$ and $X_2(s)$ their respective Laplace transform. Let R be the ROC of $X(s)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(s)$.

	Signal	L.T.	ROC	
Linearity:	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$	(40)
Time Shifting:	$x(t - t_0)$	$e^{-st_0}X(s)$	R	(41)
Shifting in s:	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + s_0$	(42)
Conjugation:	$x^*(t)$	$X^*(s^*)$	R	(43)
Time Scaling:	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	aR	(44)
Convolution:	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$	(45)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$sX(s)$	At least R	(46)
Differentiation in s:	$-\,tx(t)$	$\frac{dX(s)}{ds}$	R	(47)
Integration :	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \mathcal{R}e\{s\} > 0$	(48)

Some Laplace Transform Pairs

Signal	LT	ROC	
$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$	(49)
$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$	(50)
$u(t)\cos(\omega_0 t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > 0$	(51)
$-e^{-\alpha t}u(-t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} < -\alpha$	(52)
$\delta(t)$	1	all s	(53)

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{54}$$

Properties of z-Transform

Let $x[n]$, $x_1[n]$ and $x_2[n]$ be three DT signals and denote by $X(z)$, $X_1(z)$ and $X_2(z)$ their respective z-transform. Let R be the ROC of $X(z)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(z)$.

	Signal	z-T.	ROC	
Linearity:	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \cap R_2$	(55)
Time Shifting:	$x[n - n_0]$	$z^{-n_0}X(z)$	R, but perhaps adding/deleting $z = 0$	(56)
Time Shifting:	$x[-n]$	$X(z^{-1})$	R^{-1}	(57)
Scaling in z:	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R	(58)
Conjugation:	$x^*[n]$	$X^*(z^*)$	R	(59)
Convolution:	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$	(60)

Some z-Transform Pairs

Signal	LT	ROC	
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$	(61)
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$\|z\| < 1$	(62)
$\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\|z\| > \alpha$	(63)
$-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\|z\| < \alpha$	(64)
$\delta[n]$	1	all z	(65)