EE301 Signals and Systems
Homework #1 Solutions

1.48

b) $z_2 = r_0$,
$$\begin{cases} x_0 = r_0 \\ y_0 = 0. \end{cases}$$

c) $z_3 = r_0 e^{j(\theta_0 + \pi)}$
$$\begin{cases} x_0 = r_0 \cos(\theta_0 + \pi) = -r_0 \cos\theta_0 \\ y_0 = r_0 \sin(\theta_0 + \pi) = -r_0 \sin\theta_0 . \end{cases}$$

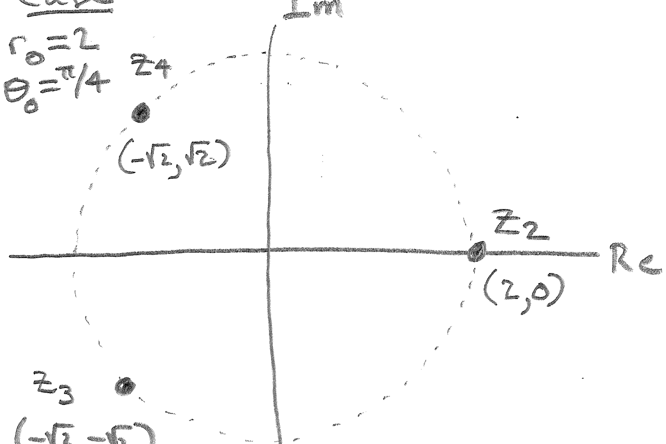
⊛ Note that $\begin{bmatrix} \cos(\theta + \pi) = -\cos\theta \\ \sin(\theta + \pi) = -\sin\theta \end{bmatrix}$

d) $z_4 = r_0 e^{j(-\theta_0 + \pi)}$
$$\begin{cases} x_0 = r_0 \cos(-\theta_0 + \pi) = -r_0 \cos(-\theta_0) = -r_0 \cos\theta_0 . \\ y_0 = r_0 \sin(-\theta_0 + \pi) = -r_0 \sin(-\theta_0) = r_0 \sin\theta_0 . \end{cases}$$

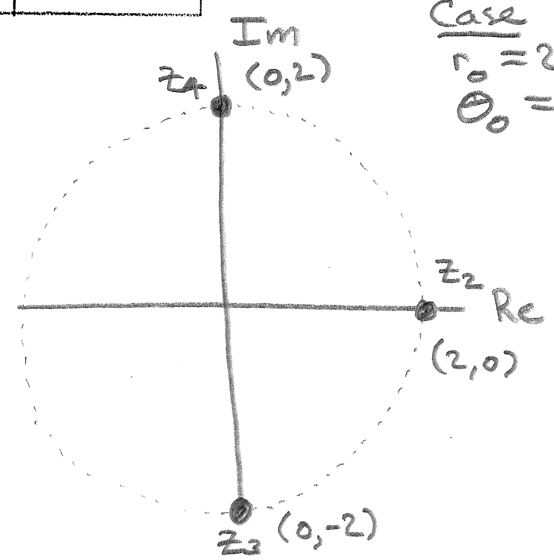
⊛ Note that $\begin{bmatrix} \cos(-\theta_0) = \cos\theta_0 \\ \sin(-\theta_0) = -\sin\theta_0 \end{bmatrix}$.

r_0, θ_0 \ z_n	z_2	z_3	z_4
$r_0 = 2$, $\theta_0 = \frac{\pi}{4}$	2	$2e^{j\frac{5}{4}\pi} = -\sqrt{2} - j\sqrt{2}$	$2e^{j\frac{3}{4}\pi} = -\sqrt{2} + j\sqrt{2}$
$r_0 = 2$, $\theta_0 = \frac{\pi}{2}$	2	$2e^{j\frac{3}{2}\pi} = -2j$	$2e^{j\frac{\pi}{2}} = 2j$

Case $r_0 = 2$, $\theta_0 = \pi/4$



Case $r_0 = 2$, $\theta_0 = \pi/2$



EE301 Signals and Systems
Homework #1 Solutions

$\boxed{1.49}$ Formulas to be used:

$$r_0 = \sqrt{x_0^2 + y_0^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y_0}{x_0}\right), & x_0 > 0, \\ \pi - \tan^{-1}\left(\frac{y_0}{x_0}\right), & x_0 < 0, y_0 > 0, \\ -\pi - \tan^{-1}\left(\frac{y_0}{x_0}\right), & x_0 \leq 0, y_0 \leq 0. \end{cases}$$

c) $-5 - 5j = 5\sqrt{2}\, e^{-j\frac{3}{4}\pi}$

g) $(\sqrt{3} + j^3)(1 - j) = (\sqrt{3} - j)(1 - j)$

$$= 2e^{-j\frac{\pi}{6}} \cdot \sqrt{2} \cdot e^{-j\frac{\pi}{4}}$$

$$= 2\sqrt{2}\, e^{-j\frac{5}{12}\pi}$$

k) $(\sqrt{3} + j) \cdot 2\sqrt{2}\, e^{-j\frac{\pi}{4}}$

$$= 2e^{j\frac{\pi}{6}} \cdot 2\sqrt{2} \cdot e^{-j\frac{\pi}{4}}$$

$$= 4\sqrt{2}\, e^{-j\frac{\pi}{12}}$$

l) $\dfrac{e^{j\frac{\pi}{3}} - 1}{1 + j\sqrt{3}} = \dfrac{e^{j\frac{\pi}{3}} - 1}{2e^{j\frac{\pi}{3}}} = \frac{1}{2}\left(1 - e^{-j\frac{\pi}{3}}\right)$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2} e^{j\frac{\pi}{3}}$$

EE301 Signals and Systems
Homework #1 Solutions

1.51 Euler's relation : $e^{j\theta} = \cos\theta + j\sin\theta$

d) by utilizing identies in O&W 1.51 (a) & (b).

$$(\sin\theta)(\sin\phi) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \cdot \frac{1}{2j}(e^{j\phi} - e^{-j\phi})$$
$$= -\frac{1}{4}[e^{j(\theta+\phi)} - e^{j(\phi-\theta)} - e^{j(\theta-\phi)} + e^{-j(\theta+\phi)}]$$
$$= -\frac{1}{2}\left[\frac{e^{j(\theta+\phi)} + e^{-j(\theta+\phi)}}{2} - \frac{e^{j(\theta-\phi)} + e^{-j(\theta-\phi)}}{2}\right]$$
$$= \frac{1}{2}\cos(\theta-\phi) - \frac{1}{2}\cos(\theta+\phi)$$

c) $(e^{j\theta})^2 = (\cos\theta + j\sin\theta)^2$

$$e^{j2\theta} = \cos^2\theta + 2j\sin\theta\cos\theta + \sin^2\theta$$
$$\cos(2\theta) + j\sin(2\theta) = 2\cos^2\theta - 1 + 2j\sin\theta\cos\theta$$

Thus $\cos(2\theta) = 2\cos^2\theta - 1$

$$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)].$$

1.52

b) $\frac{z}{z^*} = \frac{re^{j\theta}}{re^{-j\theta}} = e^{j2\theta}$

d) $z - z^* = re^{j\theta} - re^{-j\theta}$

$$= r(e^{j\theta} - e^{-j\theta})$$
$$= j \cdot 2r\sin\theta$$
$$\mathrm{Im}\{z\} = \mathrm{Im}\{re^{j\theta}\}$$
$$= \mathrm{Im}\{r\cos\theta + j\cdot r\sin\theta\}.$$
$$= r\sin\theta.$$

EE301 Signals and Systems
Homework #1 Solutions

h) Making use of the identities in O&W 1.52(c) and (g).

$$\text{Re}\left\{\frac{z_1}{z_2}\right\} = \frac{1}{2}\left\{\frac{z_1}{z_2} + \left(\frac{z_1}{z_2}\right)^*\right\}.$$

$$= \frac{1}{2}\left\{\frac{z_1}{z_2} + \frac{z_1^*}{z_2^*}\right\}.$$

$$= \frac{1}{2}\left[\frac{z_1 z_2^* + z_1^* z_2}{z_2 z_2^*}\right].$$

Alternatively you could plug $\begin{cases} z_1 = r_1 e^{j\theta_1} \\ z_2 = r_2 e^{j\theta_2} \end{cases}$ in both sides of the equation and see whether they are indeed equal.

$\boxed{1.54}$

a) Finite Sum Formula. $\displaystyle\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1. \end{cases}$

Proof: Let $\displaystyle\sum_{n=0}^{N-1} \alpha^n = P$

i) $\alpha = 1$, $P = \underbrace{1 + 1 + \cdots + 1}_{N} = N.$

ii) $\alpha \neq 1$,

$$\alpha \cdot P = \alpha \cdot \sum_{n=0}^{N-1} \alpha^n$$

$$= \alpha \cdot (1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1})$$

$$= \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{N-1} + \alpha^N$$

$$\alpha \cdot P + 1 = 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N$$

$$= P + \alpha^N$$

Thus $P = \frac{1-\alpha^N}{1-\alpha}.$

EE301 Signals and Systems
Homework #1 Solutions

b) Infinite Sum Formula $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$, $|\alpha|<1$

Proof : Let $\sum_{n=0}^{\infty} \alpha^n = Q$.

$$\alpha \cdot Q = \alpha \cdot \sum_{n=0}^{\infty} \alpha^n = \alpha(1+\alpha+\alpha^2+\cdots)$$
$$= \alpha+\alpha^2+\alpha^3+\cdots$$
$$= \sum_{n=0}^{\infty} \alpha^n - 1.$$
$$= Q-1.$$

$$Q = \frac{1}{1-\alpha}.$$

Alternatively, you could take the limit of 1.54(a).

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N\to\infty} \frac{1-\alpha^N}{1-\alpha}.$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

c) Let $\sum_{n=0}^{\infty} n\alpha^n = M$. $|\alpha|<1$

$$M = 0+\alpha+2\alpha^2+3\alpha^3+\cdots$$
$$\alpha \cdot M = 0+\alpha^2+2\alpha^3+3\alpha^4+\cdots$$
$$M-\alpha M = \alpha+\alpha^2+\alpha^3+\cdots$$
$$(1-\alpha)M = -1+1+\alpha+\alpha^2+\alpha^3+\cdots$$
$$= -1+\sum_{n=0}^{\infty} \alpha^n$$
$$= -1+\frac{1}{1-\alpha} \qquad |\alpha|<1.$$

$$M = \frac{\alpha}{(1-\alpha)^2}, \quad |\alpha|<1.$$

Alternatively,

Let $F(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x|<1$

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$
$$= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) \quad \circledast$$
$$= \sum_{n=0}^{\infty} n x^{n-1}.$$

$$\therefore \sum_{n=0}^{\infty} n x^n = x \cdot \frac{d}{dx} F(x)$$
$$= x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right)$$
$$= \frac{x}{(1-x)^2}.$$

Substituting x with α.

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}, \quad |\alpha|<1$$

EE301 Signals and Systems
Homework #1 Solutions

d) $\sum_{n=k}^{\infty} \alpha^n$

$$= \sum_{n=0}^{\infty} \alpha^n - \sum_{n=0}^{k-1} \alpha^n$$

$$= \frac{1}{1-\alpha} - \frac{1-\alpha^k}{1-\alpha}$$

$$= \frac{\alpha^k}{1-\alpha}, \quad |\alpha| < 1.$$

$\boxed{1.55}$ d) $\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n e^{j\pi n/2}$

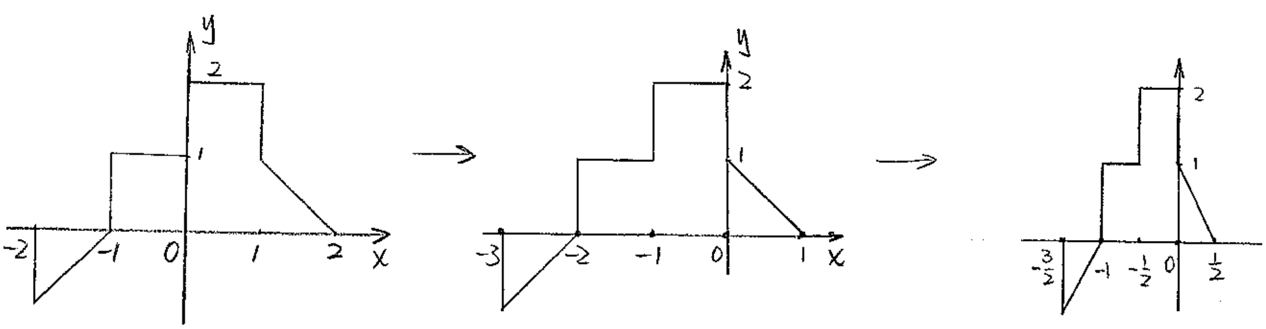
$$= \sum_{n=2}^{\infty} \left(\frac{1}{2} e^{j\frac{\pi}{2}}\right)^n$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2} j\right)^n$$

$$= \frac{\left(\frac{1}{2}j\right)^2}{1-\frac{1}{2}j} = \frac{-\frac{1}{4}}{1-\frac{1}{2}j} = \frac{-\frac{1}{4}\cdot\left(1+\frac{1}{2}j\right)}{\frac{5}{4}} = -\frac{1+\frac{1}{2}j}{5} = -\frac{1}{5} - j\frac{1}{10}.$$

$\boxed{1.21}$

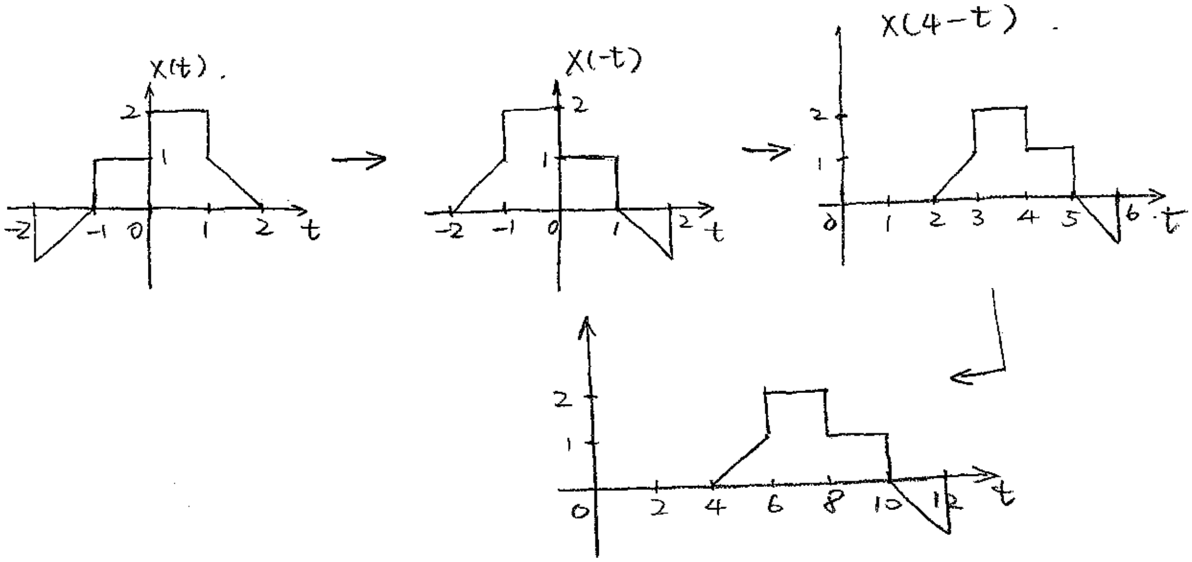
c) $x(t) \xrightarrow{\text{shift to left}} x(t+1) \xrightarrow{\text{compress by 2}} x(2t+1).$



EE301 Signals and Systems
Homework #1 Solutions

d). $x(4-\frac{t}{2})$

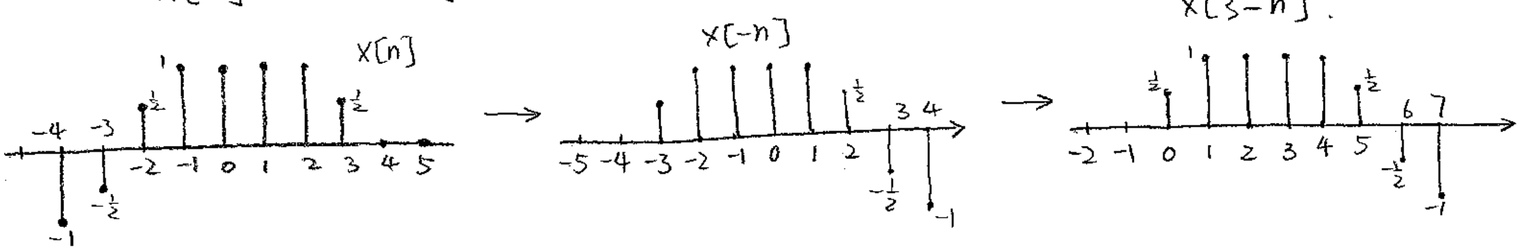
$x(t) \rightarrow x(-t) \rightarrow x(4-t) \rightarrow x(4-\frac{t}{2})$.



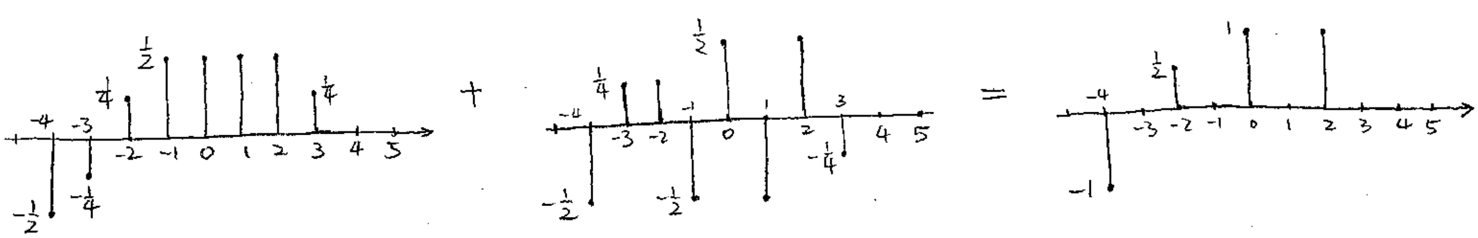
$\boxed{1.22}$

b) $x[3-n]$.

$x[n] \rightarrow x[-n] \rightarrow x[3-n]$.



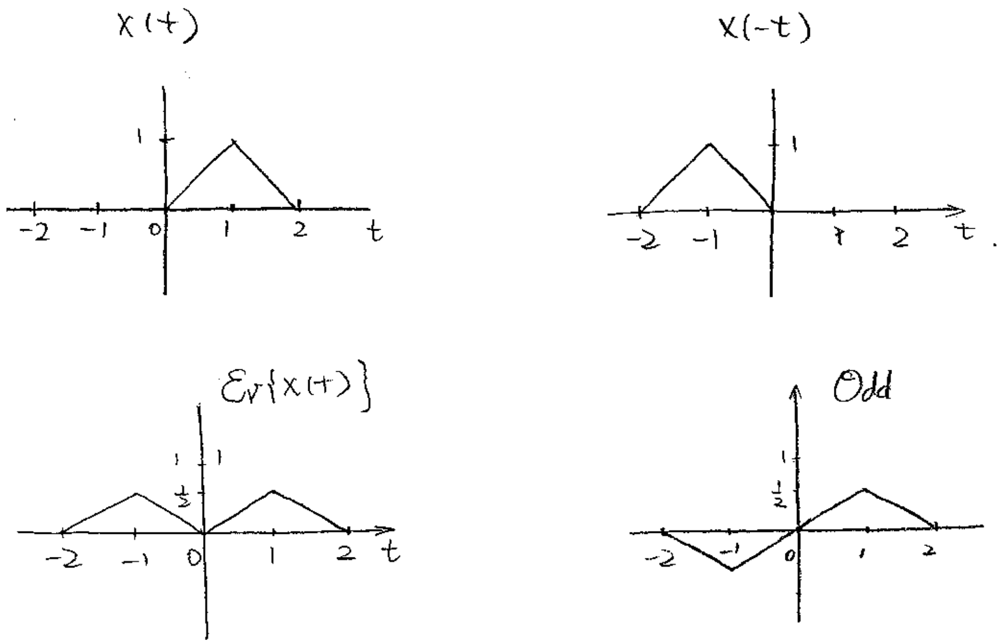
g) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$.



EE301 Signals and Systems
Homework #1 Solutions

1.23 a). $\mathcal{E}v\{x(t)\} = \dfrac{x(t)+x(-t)}{2}$,

$\mathcal{O}dd\{x(t)\} = \dfrac{x(t)-x(-t)}{2}$.



1.26 a) $x[n+7] = \sin\left[\frac{6\pi}{7}(n+7)+1\right]$

$= \sin\left(\frac{6\pi}{7}n + 6\pi + 1\right)$

$= \sin\left(\frac{6\pi}{7}n + 1\right)$

$= x[n]$.

Thus $x[n]$ is periodic with a period of 7.

c). Suppose $x[n+N] = x[n]$.

$\cos\left[\frac{\pi}{8}(n+N)^2\right] = \cos\left(\frac{\pi}{8}n^2\right)$.

$\cos\left(\frac{\pi}{8}n^2 + \frac{\pi}{4}n\cdot N + \frac{\pi}{8}N^2\right) = \cos\left(\frac{\pi}{8}n^2\right)$.

When $\frac{\pi}{4}\cdot n\cdot N + \frac{\pi}{8}N^2$ is a multiple of π, the signal is periodic.

The smallest N to achieve this would be 8.

EE301 Signals and Systems
Homework #1 Solutions

9/9

$\boxed{1.32}$ 3) True

4) True

For both problems, the period of $y_2(t)$ is twice that of $x(t)$.

$\boxed{1.33}$ 3) True

4) True.

For both problems, the period of $y_2[n]$ is twice that of $x[n]$.

See following pages for a complete solution of 1.33

O+W 1.33 (3, 4)

$x[n]$ a DT signal. Define another DT signal by

$$y_2[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Prove or disprove:

(3) If $x[n]$ is periodic then $y_2[n]$ is periodic

(4) If $y_2[n]$ is periodic then $x[n]$ is periodic

Also find relationships between periods of $x[\cdot]$ and $y_2[\cdot]$ when appropriate.

$\boxed{\text{Re (3)}}$ Suppose $x[n]$ is periodic ie $x[n] = x[n+N]$ for all $n \in \mathbb{Z}$. Look at N even and N odd separately

$N = 2N_0$ ie even

$y_2[0] = x[0]$
$y_2[1] = 0$
$y_2[2] = x[1]$
$\vdots$
$y_2[N-1] = 0$
$y_2[N] = x[N/2] = x[N_0]$
$\vdots$
$y_2[2N-2] = x[N-1]$
$y_2[2N-1] = 0$
$y_2[2N] = x[N] = x[0]$

$\Longrightarrow y_2[\cdot]$ is periodic and its period is $2N$

$\therefore$ (3) holds.

$N = 2N_0 + 1$ ie odd

$y_2[0] = x[0]$
$y_2[1] = 0$
$y_2[2] = x[1]$
$\vdots$
$y_2[N-1] = x[N_0]$
$y_2[N] = 0$
$\vdots$
$y_2[2N-4] = x[N-2]$
$y_2[2N-3] = 0$
$y_2[2N-2] = x[N-1]$
$y_2[2N-1] = 0$
$y_2[2N] = x[N] = x[0]$

$\Longrightarrow y_2[\cdot]$ is periodic & its period is $2N$

$\boxed{\text{Re }(4)}$ Suppose that $y_2[\cdot]$ is periodic ie $y_2[n] = y_2[n+N]$ for all $n \in \mathbb{Z}$ and some N a positive integer. Again we look at the separate cases.

<u>$N = 2N_0$ ie even</u>

$y_2[0] = x[0]$
$y_2[1] = 0$
$y_2[2] = x[1]$
$\vdots$
$y_2[N-1] = 0$
$y_2[N] = x[N/2] = x[N_0] = y_2[0] = x[0]$
$y_2[N+1] = 0 = y_2[1]$
$y_2[N+2] = x[\frac{N}{2}+1] = y_2[2] = x[1]$
$\vdots$

Then easily see that $x[\cdot]$ must be periodic and of period $N/2$ (a pos. integer in this case so it makes sense).

<u>$N = 2N_0 + 1$ ie odd</u>

$y_2[0] = x[0]$
$y_2[1] = 0$
$y_2[2] = x[1]$
$\vdots$
$y_2[N-2] = y_2[2N_0 - 1] = 0$
$y_2[N-1] = y_2[2N_0] = x[N_0]$
$y_2[N] = y_2[2N_0+1] = 0 = y_2[0] \Longrightarrow$ ie $x[0] = 0$
$y_2[N+1] = y_2[2N_0+2] = x[N_0+1]$
$= y_2[1] = 0 \Longrightarrow x[N_0+1] = 0$

$$y_2[N+2] = y_2[2N_0+3] = 0 \implies x[1] = 0$$
$$= y_2[2] = x[1]$$

$\vdots$

$$x[n] = 0 \quad \forall n \implies y_2[n] = 0 \quad \forall n.$$

$\therefore$ If $y_2[\cdot]$ is periodic with odd period and created from a signal $x[\cdot]$ as prev. described then both signals must be identically zero.