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ECE 301 Signals and Systems
Homework # 7 Solution

5.21 d)

$$x[n] = 2^n \sin(\tfrac{\pi}{4}n) u[-n]$$
$$= (\tfrac{1}{2})^{-n} u[-n] \left(\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j} \right)$$

Let $x_0[n] = (\frac{1}{2})^n u[n]$. then $x[n] = x_0[-n] \left(\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j} \right)$

By Table 5.1.

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$
$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$$

$$X(e^{j\omega}) = \frac{1}{2j}\left[X_0(e^{-j(\omega-\frac{\pi}{4})}) - X_0(e^{-j(\omega+\frac{\pi}{4})})\right].$$

By Table 5.2. $X_0(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$X(e^{j\omega}) = \frac{1}{2j}\left[\frac{1}{1-\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{1-\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}}\right]$$

j)

$$x[n] = (n-1)(\tfrac{1}{3})^{|n|}$$
$$= (n-1)(\tfrac{1}{3})^n u[n] + (n-1)(\tfrac{1}{3})^{-n} u[-n-1].$$
$$= (n+1)(\tfrac{1}{3})^n u[n] - 2\cdot(\tfrac{1}{3})^n u[n]$$
$$- (-n+1)(\tfrac{1}{3})^{-n} u[-n] + 1.$$

$$\therefore X(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}e^{-j\omega})^2} - \frac{2}{1-\frac{1}{3}e^{-j\omega}} - \frac{1}{1-\frac{1}{3}e^{j\omega}} + 1$$

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ECE 301 Signals and Systems
Homework # 7 Solution

5.22 d) $X(e^{jw}) = \cos^2 w + \sin^2 3w$

$$= \frac{1+\cos 2w}{2} + \frac{1-\cos 6w}{2}$$

$$= 1 + \frac{1}{2}\cos 2w - \frac{1}{2}\cos 6w.$$

$$= 1 + \frac{1}{4}e^{j2w} + \frac{1}{4}e^{-j2w} - \frac{1}{4}e^{j6w} - \frac{1}{4}e^{-j6w}.$$

By Table 5.2

$$\delta[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-jwn_0}$$

$$x[n] = 1 + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-6] - \frac{1}{4}\delta[n+6].$$

h) $X(e^{jw}) = \frac{1-(\frac{1}{3})^6 e^{-j6w}}{1-\frac{1}{3}e^{-jw}}.$

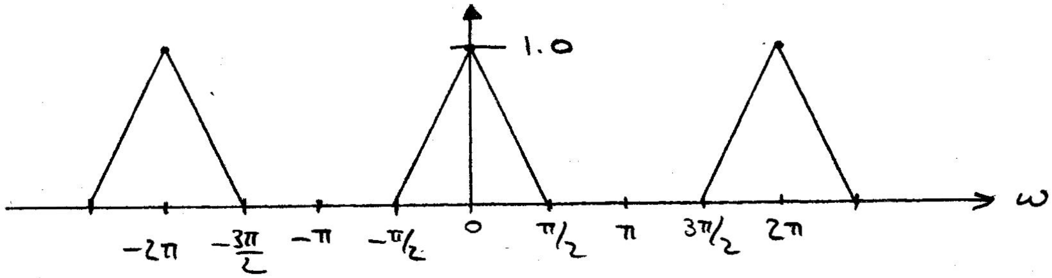
$$= 1 + (\tfrac{1}{3})e^{-jw} + (\tfrac{1}{3})^2 e^{-j2w} + (\tfrac{1}{3})^3 e^{-j3w} + (\tfrac{1}{3})^4 e^{-j4w} + (\tfrac{1}{3})^5 e^{-j5w}$$

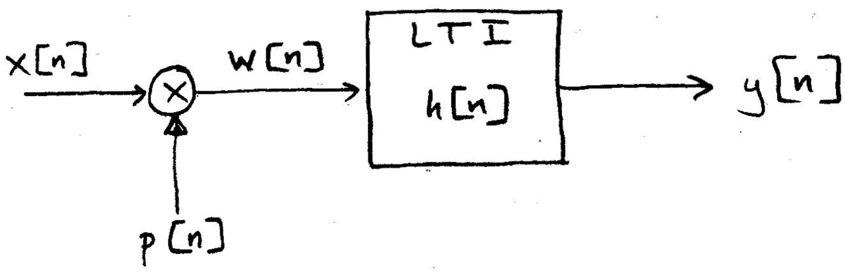
By the same transformation pair used in part (d),

$$x[n] = 1 + (\tfrac{1}{3})\delta[n-1] + (\tfrac{1}{3})^2\delta[n-2] + (\tfrac{1}{3})^3\delta[n-3] + (\tfrac{1}{3})^4\delta[n-4] + (\tfrac{1}{3})^5\delta[n-5].$$

O+W 5.27 (a,b) for the p[n] defined in ii and iv

$x[n] \longleftrightarrow X(e^{j\omega})$





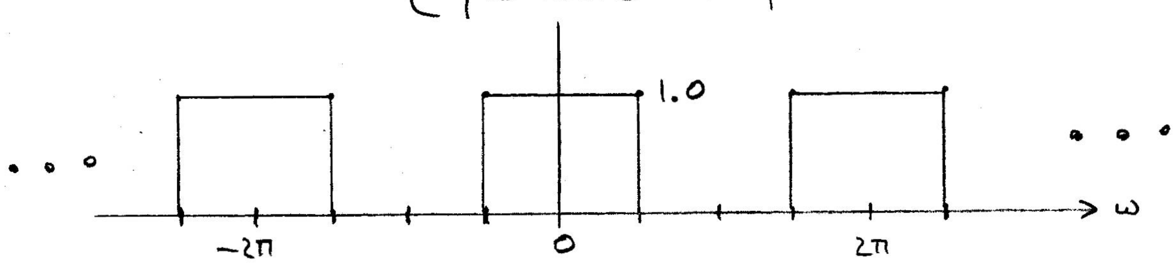
where

$$h[n] = \frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2}\frac{\sin(\pi n/2)}{\pi n/2}$$

(See Table 5.2)

$$\updownarrow$$

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/2 \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \\ \text{periodic of period } 2\pi \end{cases}$$



(a) Find $W(e^{j\omega})$ and (b) $y[n]$ for two cases of $p[n]$:

(ii) $p[n] = \cos(\pi n/2)$

(iv) $p[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$

(ii) $p[n] = \cos(\pi n/2) = \frac{1}{2} e^{-j\pi n/2} + \frac{1}{2} e^{+j\pi n/2}$

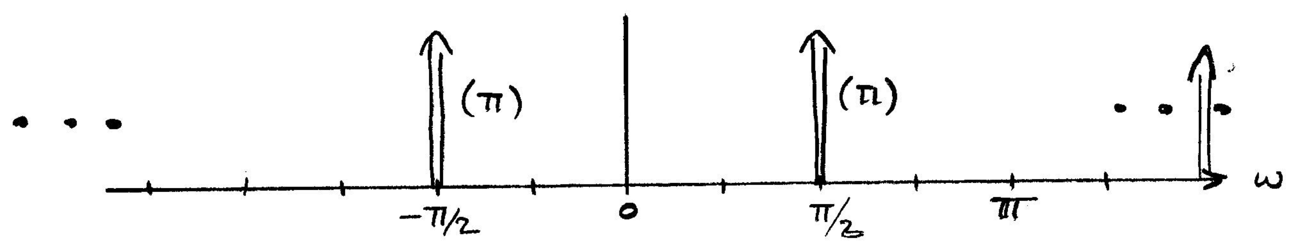
$= \frac{1}{2} e^{-j2\pi n/4} + \frac{1}{2} e^{+j2\pi n/4}$

So $p[n]$ is periodic of period 4 and its DTFS coeffs are

$$P_{-1} = P_1 = \frac{1}{2} \quad \text{and} \quad P_0 = P_2 = 0$$

and periodic of period 4. The corresponding DTFT is

$$P(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} P_k \, \delta\left(\omega - \frac{2\pi k}{4}\right)$$

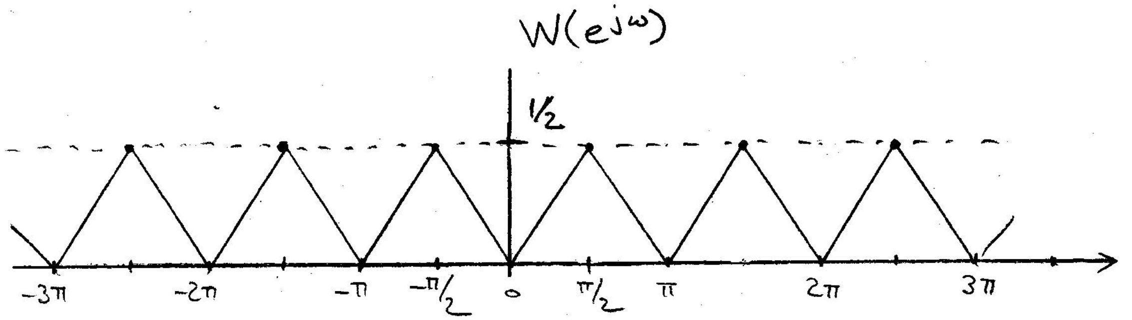


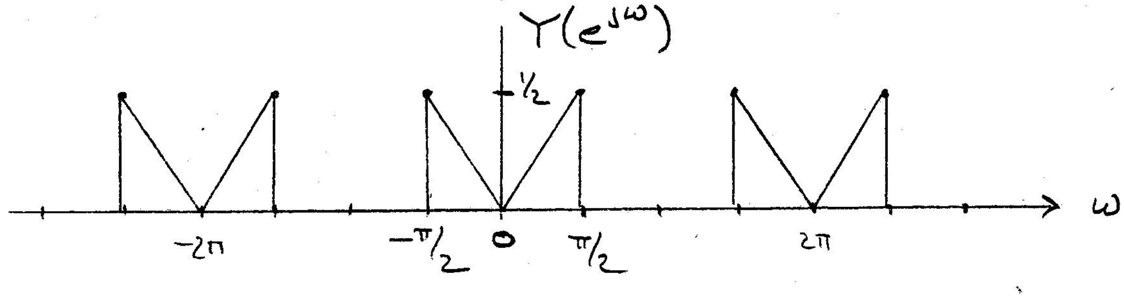
From the mult. in time ⟷ conv. in freq prop

$$W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) \, d\theta$$

$$= \frac{1}{2\pi} \cdot \pi \int_{-\pi}^{\pi} \left[\delta\left(\theta + \frac{\pi}{2}\right) + \delta\left(\theta - \frac{\pi}{2}\right)\right] X(e^{j(\omega-\theta)}) \, d\theta$$

$$= \frac{1}{2} X(e^{j(\omega+\pi/2)}) + \frac{1}{2} X(e^{j(\omega-\pi/2)})$$





For (b) we want to compute $y[n] \longleftrightarrow Y(e^{j\omega})$ for which we go directly to the inverse transform integral

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Now

$$\begin{aligned} Y(e^{j\omega}) &= \tfrac{1}{\pi}\omega && \text{for } 0 \le \omega < \pi/2 \\ &= -\tfrac{1}{\pi}\omega && \text{for } -\pi/2 < \omega \le 0 \\ &= 0 && \text{for the remainder of the period of length } 2\pi \end{aligned}$$

$$\Longrightarrow\ y[n] = \frac{1}{2\pi}\int_{-\pi/2}^{0}\left(-\tfrac{1}{\pi}\omega\right)e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0}^{\pi/2}\left(\tfrac{1}{\pi}\omega\right)e^{j\omega n}\,d\omega$$

From the integration by parts formula we can conclude that

$$\int \omega e^{j\omega n}\,d\omega = \frac{\omega}{jn}e^{j\omega n} - \left(\frac{1}{jn}\right)^2 e^{j\omega n}$$

Thus

$$\int_{-\pi/2}^{0} \omega e^{j\omega n}\, d\omega = \frac{1}{jn}\left[\omega - \frac{1}{jn}\right] e^{j\omega n}\Big|_{\omega=-\pi/2}^{0}$$

$$= -\left(\frac{1}{jn}\right)^2 - \frac{1}{jn}\left[-\frac{\pi}{2} - \frac{1}{jn}\right] e^{-j\pi n/2}$$

$$= \frac{1}{n^2} + \left[\frac{\pi}{j2n} - \frac{1}{n^2}\right] e^{-j\pi n/2} \qquad (*_1)$$

$$\int_{0}^{\pi/2} \omega e^{j\omega n}\, d\omega = \frac{1}{jn}\left[\omega - \frac{1}{jn}\right] e^{j\omega n}\Big|_{\omega=0}^{\pi/2}$$

$$= \frac{1}{jn}\left[\frac{\pi}{2} - \frac{1}{jn}\right] e^{j\pi n/2} + \left(\frac{1}{jn}\right)^2$$

$$= \left[\frac{\pi}{j2n} + \frac{1}{n^2}\right] e^{j\pi n/2} - \frac{1}{n^2} \qquad (*_2)$$

$$\therefore\ y[n] = -\frac{1}{2\pi^2}\cdot (*_1) + \frac{1}{2\pi^2}\cdot (*_2)$$

$$= -\frac{1}{2\pi^2}\frac{1}{n^2} - \frac{1}{2\pi^2}\left[\frac{\pi}{j2n} - \frac{1}{n^2}\right] e^{-j\pi n/2} + \frac{1}{2\pi^2}\left[\frac{\pi}{j2n} + \frac{1}{n^2}\right] e^{j\pi n/2}$$

$$- \frac{1}{2\pi^2}\frac{1}{n^2}$$

$$= \frac{1}{2\pi^2 n^2}\left[e^{j\pi n/2} + e^{-j\pi n/2}\right] + \frac{1}{2\pi^2}\cdot\frac{\pi}{j2n}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right]$$

$$= \frac{\cos(\pi n/2)}{\pi^2 n^2} + \frac{1}{2\pi n}\sin(\pi n/2) - \frac{1}{\pi^2 n^2}$$

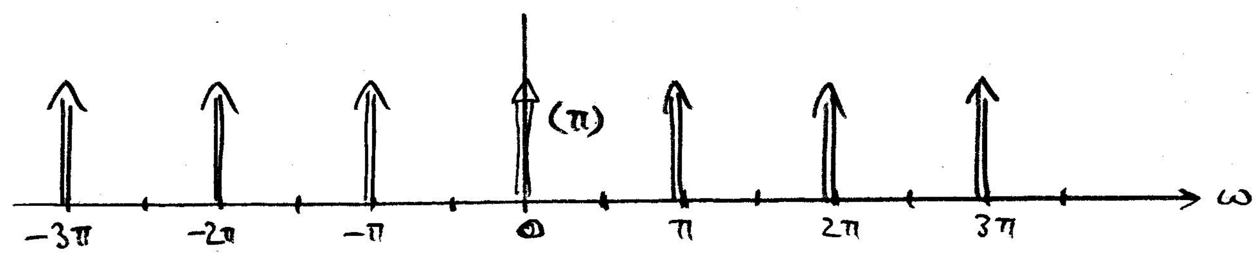
$$\therefore \; y[n] = \frac{\sin(\pi n/2)}{2\pi n} - \frac{1-\cos(\pi n/2)}{\pi^2 n^2}$$

(for b-ii)

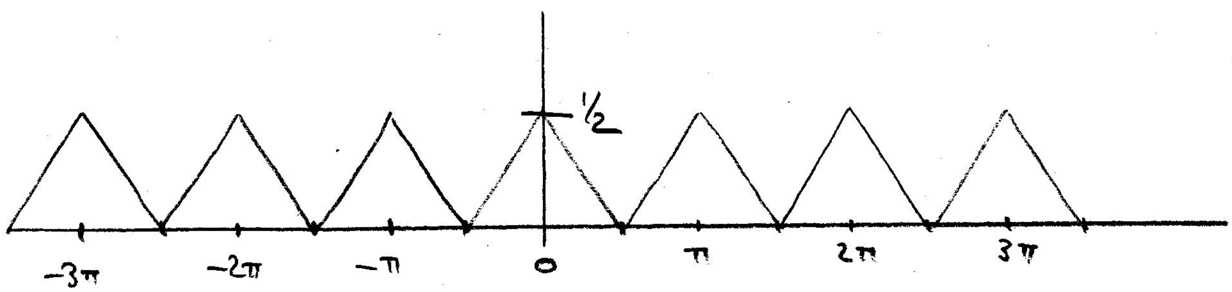
(iv) $p[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$

$\updownarrow$ (Table 5.2)

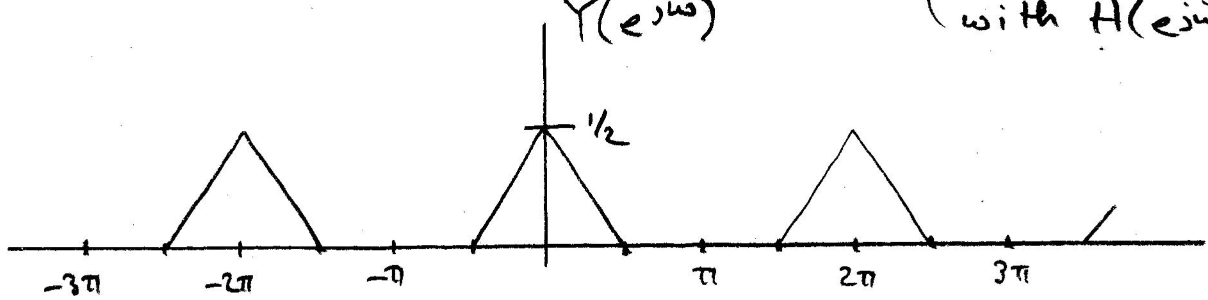
$$P(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)$$



$W(e^{j\omega})$ (via convolution)



$Y(e^{j\omega})$ (via filtering with $H(e^{j\omega})$)



Finally we compute

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\, e^{j\omega n}\, d\omega$$

using integration by parts (basically same formula as before) to get

$$y[n] = 2\left[\frac{\sin(\pi n/4)}{\pi n}\right]^2$$

ECE 301 Signals and Systems
Homework # 8 Solution

5.29 a) ii $x[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}$

$h[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$\therefore Y(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})^2} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{1}{(1-\frac{1}{4}e^{-j\omega})^2} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$y[n] = 4(\tfrac{1}{2})^n u[n] - (n+1)(\tfrac{1}{4})^n u[n] - 2(\tfrac{1}{4})^n u[n].$$

b) i. $x[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$h[n] = \left[(\tfrac{1}{2})^n \cos(\tfrac{\pi n}{2})\right] u[n]$$

$$= (\tfrac{1}{2})^n u[n] \cdot \cos(\tfrac{\pi n}{2})$$

$$= (\tfrac{1}{2})^n u[n] \left(\frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}\right)$$

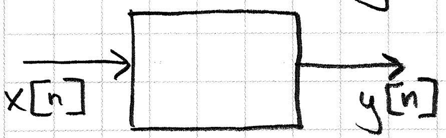
$$H(e^{j\omega}) = \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}e^{-j(\omega+\frac{\pi}{2})}} + \frac{1}{1-\frac{1}{2}e^{-j(\omega-\frac{\pi}{2})}}\right].$$

$$Y(e^{j\omega}) = \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}e^{-j(\omega+\frac{\pi}{2})}} + \frac{1}{1-\frac{1}{2}e^{-j(\omega-\frac{\pi}{2})}}\right] \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[n] = \begin{cases} (\frac{1}{2})^n, & n = 4k+2,\ 4k+3, \quad k = 0, 1, 2, \dots \\ 0, & \text{elsewhere.} \end{cases}$$

O+W 5.35

Causal LTI syst. has diff. eqn. model



$$y[n] - a\,y[n-1] = b\,x[n] + x[n-1]$$

where $a \in \mathbb{R}$ and $|a| < 1$.

(a) Find a value of b s.t.

$$|H(e^{j\omega})| = 1 \quad \forall \omega$$

ie an <u>all-pass system</u>.

Taking the DTFT of the difference equation and using shifting property

$$Y(e^{j\omega}) - a e^{-j\omega} Y(e^{j\omega}) = b X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega})$$

$$\Longrightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}}$$

Then

$$|H(e^{j\omega})|^2 = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}} \cdot \frac{b + e^{+j\omega}}{1 - a e^{+j\omega}}$$

$$= \frac{b^2 + b(e^{j\omega} + e^{-j\omega}) + 1}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2}$$

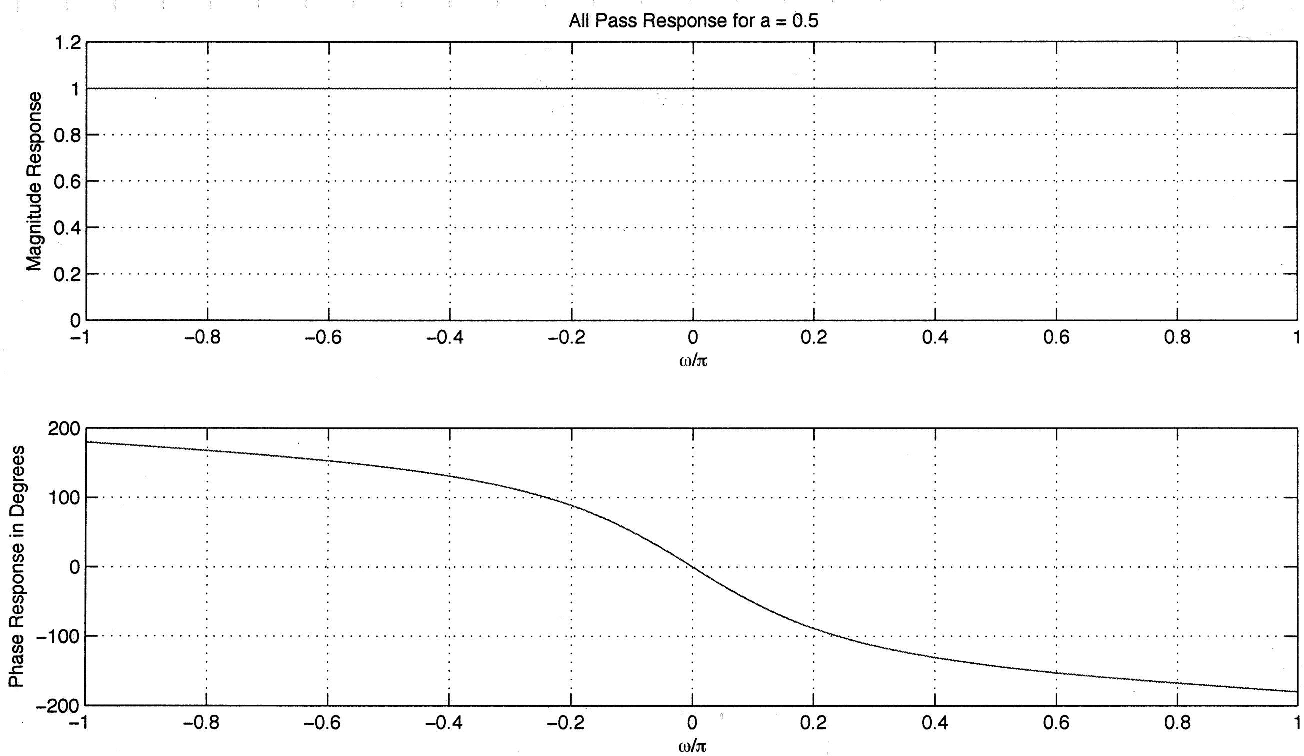
$$= \frac{1 + b^2 + 2b\cos(\omega)}{1 + a^2 - 2a\cos(\omega)} \overset{\text{Set}}{=} 1 \quad \forall \omega$$

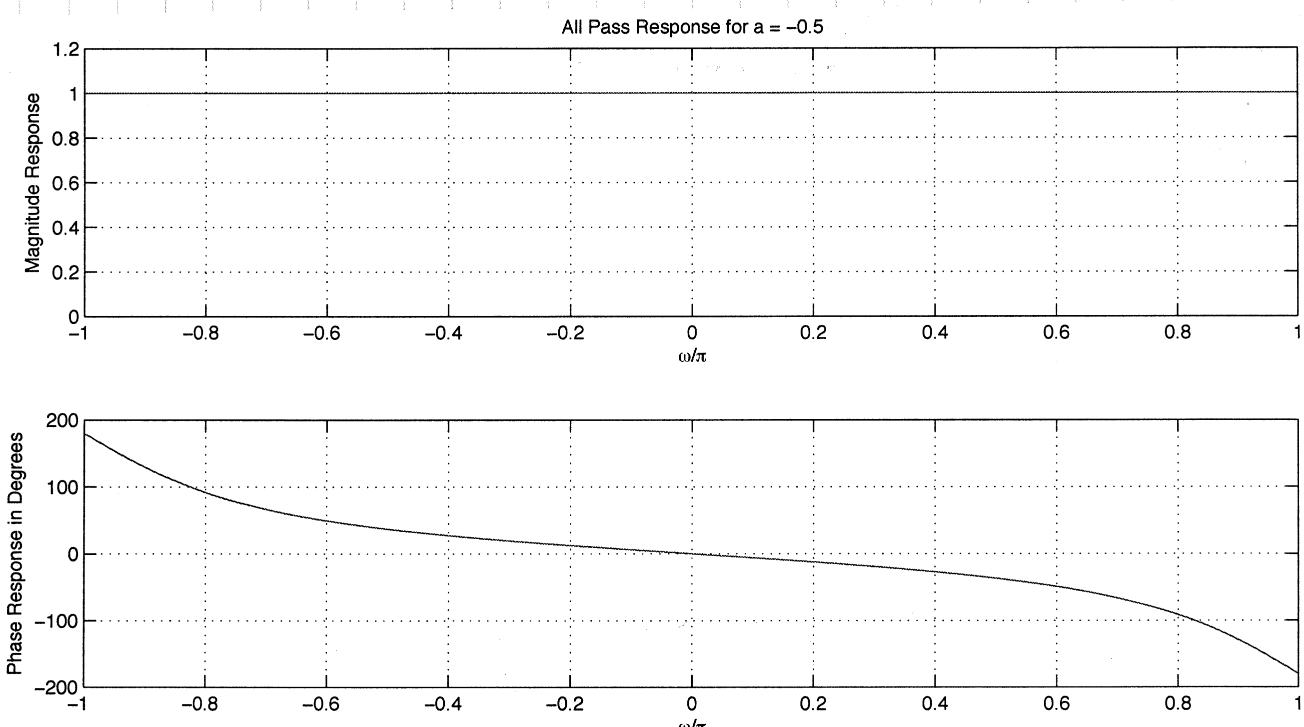
$$\therefore\ b = -a \Longrightarrow H(e^{j\omega}) = \frac{-a + e^{-j\omega}}{1 - a e^{-j\omega}}$$

(b) Sketch $\angle H(e^{j\omega})$ $0 \le \omega \le \pi$ for $a = 1/2$

(c) " " " " $a = -1/2$

The easiest way to do this would be to use Matlab.





(d) Find and plot the output of this system when $a = -1/2$ when the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

One way to solve is to use transform techniques:

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$ (from Table or as done in class)

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$= \frac{-a + e^{-j\omega}}{1 - ae^{-j\omega}} \; \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

The character of the denominator term is different depending on $a = +1/2$ or $a = -1/2$.

If $a = -1/2$ (as assumed here) then denom. factorization is into distinct terms:

$$Y(e^{j\omega}) = \frac{1/2 + e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \; \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$ let $v = e^{-j\omega}$

$$= \frac{\frac{1}{2} + v}{(1 + \frac{1}{2}v)(1 - \frac{1}{2}v)} = \frac{A}{1 + \frac{1}{2}v} + \frac{B}{1 - \frac{1}{2}v}$$

where

$$A = \left.\frac{\frac{1}{2} + v}{1 - \frac{1}{2}v}\right|_{v=-2} = \frac{-3/2}{2} = -\frac{3}{4}$$

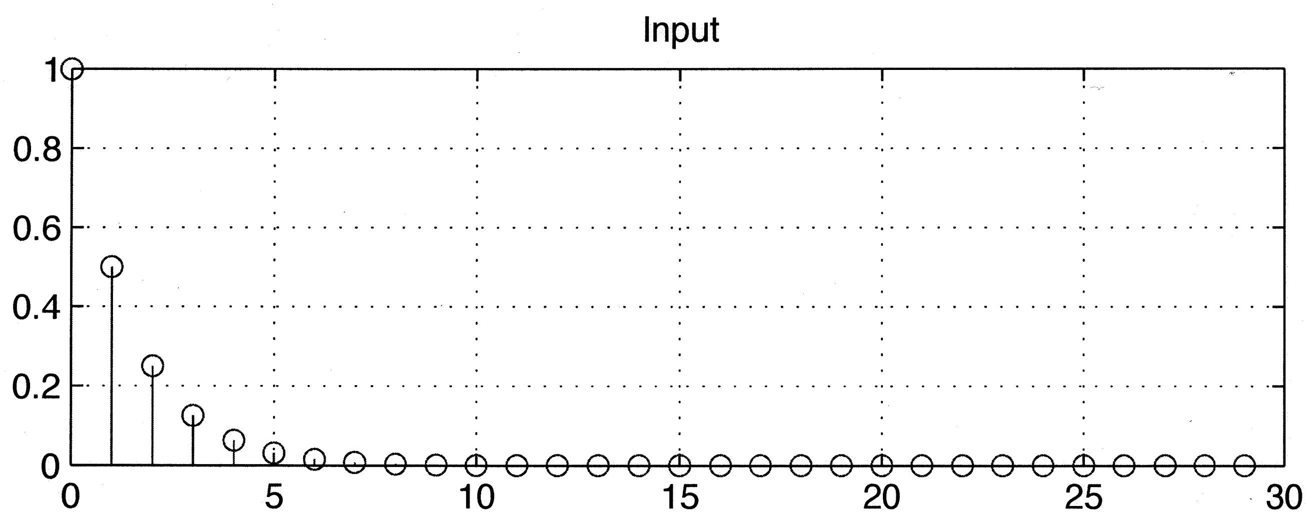
$$B = \left.\frac{\frac{1}{2} + v}{1 + \frac{1}{2}v}\right|_{v=2} = \frac{5/2}{2} = \frac{5}{4}$$

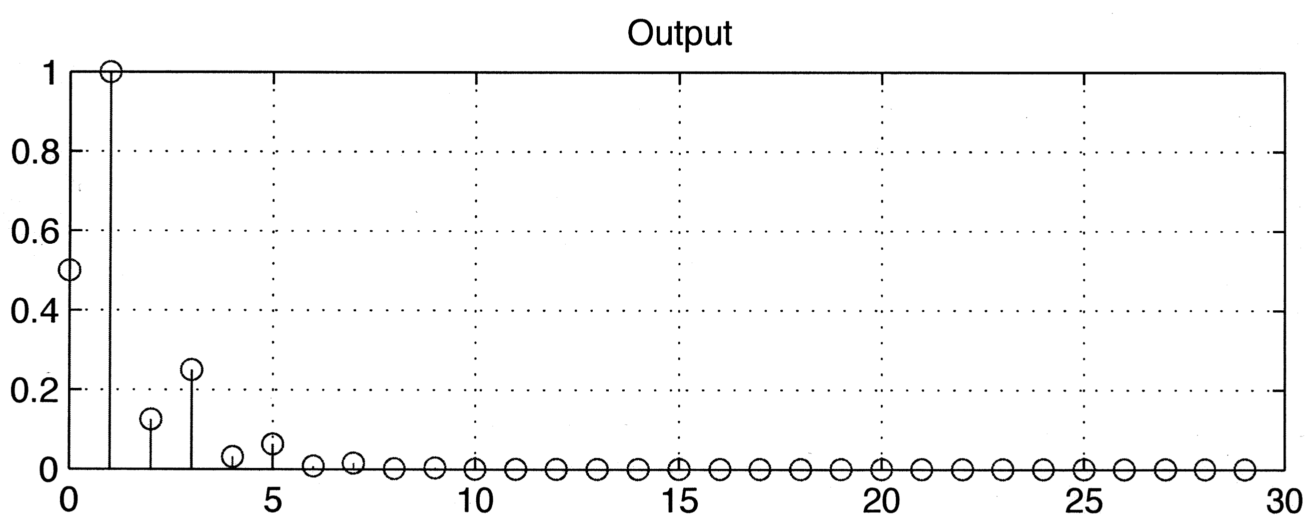
$$Y(e^{j\omega}) = \frac{-3/4}{1+\frac{1}{2}e^{-j\omega}} + \frac{5/4}{1-\frac{1}{2}e^{-j\omega}}$$

$$\updownarrow$$

$$y[n] = -\frac{3}{4}\left(-\frac{1}{2}\right)^n u[n] + \frac{5}{4}\left(\frac{1}{2}\right)^n u[n]$$

$$= \frac{1}{4}\left(\frac{1}{2}\right)^n \left[5 - 3(-1)^n\right] u[n]$$





```
%% Code for plot on OW 5.35
%

tmp = 0:0.01:pi;
L = length(tmp);
omega = [fliplr(-tmp) tmp(2:L)];

a = 0.5;

num = -a + exp(-j*omega);
den = 1 - a*exp(-j*omega);

H = num ./ den;

figure(1)

subplot(2,1,1)
plot(omega/pi,abs(H));
title('All Pass Response for a = 0.5')
ylabel('Magnitude Response')
xlabel('\omega/\pi')
grid
subplot(2,1,2)
plot(omega/pi,(180/pi)*angle(H));
ylabel('Phase Response in Degrees')
xlabel('\omega/\pi')
grid

a = -0.5;

num = -a + exp(-j*omega);
den = 1 - a*exp(-j*omega);

H = num ./ den;

figure(2)

subplot(2,1,1)
plot(omega/pi,abs(H));
title('All Pass Response for a = -0.5')
ylabel('Magnitude Response')
xlabel('\omega/\pi')
grid
subplot(2,1,2)
plot(omega/pi,(180/pi)*angle(H));
ylabel('Phase Response in Degrees')
```

```
xlabel('\omega/\pi')
grid

%% Time domain response from part (d)

n = 0:1:29;

x = (0.5) .^ n;

y = (5 - 3*((-1) .^ n))*(.25) .* x;

figure(3)

subplot(2,1,1)
stem(n,x);
title('Input')
grid
subplot(2,1,2)
stem(n,y);
title('Output')
grid
```

O+W 5.43

$x[n] \longleftrightarrow X(e^{j\omega})$

Define $g[n] = x[2n] \quad \forall n \in \mathbb{Z}$ and let $g[n] \leftrightarrow G(e^{j\omega})$
Purpose of problem is to find a relationship between $G(e^{j\omega})$ and $X(e^{j\omega})$.

(a) $v[n] \triangleq \dfrac{e^{-j\pi n} x[n] + x[n]}{2}$

$\updownarrow$

$V(e^{j\omega})$

Find relationship between $V(e^{j\omega})$ and $X(e^{j\omega})$.

Can use the Freq. Shifting property of Table 5.1 p391

$$e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

to conclude

$$e^{-j\pi n} x[n] \longleftrightarrow X(e^{j(\omega+\pi)})$$

Then linearity of the DTFT implies

$$V(e^{j\omega}) = \frac{X(e^{j(\omega+\pi)}) + X(e^{j\omega})}{2}$$

(b) Noting that $v[n] = 0$ for n odd, show that the DTFT of $v[2n]$ is $V(e^{j\omega/2})$.

Define $y[n] = v[2n]$. Then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} v[2n] e^{-j\omega n}$$

But we also have

$$V(e^{j\omega}) = \sum_{n=-\infty}^{\infty} v[n] e^{-j\omega n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} v[n] e^{-j\omega n}$$

c.o.v. in sum $n = 2\ell$

$$= \sum_{\ell=-\infty}^{\infty} v[2\ell] e^{-j\omega 2\ell} = Y(e^{j2\omega})$$

∴ Making a c.o.v. $\omega' = 2\omega$ have

$$V(e^{j\omega'/2}) = Y(e^{j\omega'})$$

which was to be shown.

(c) Show that $x[2n] = v[2n]$ from which it follows that

$$G(e^{j\omega}) = V(e^{j\omega/2})$$

and then use (a) to find $G(e^{j\omega})$ in terms of $X(e^{j\omega})$.

<u>Re: $x[2n] = v[2n]$</u>

This is obvious since $v[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Then since

$g[n] = x[2n]$ as first defined we also have
$g[n] = v[2n]$

⟹ Part (b) tells us

$$G(e^{j\omega}) = V(e^{j\omega/2})$$

Finally from (a)

$$G(e^{j\omega}) = V(e^{j\omega/2}) = \frac{X(e^{j(\frac{\omega}{2}+\pi)}) + X(e^{j\frac{\omega}{2}})}{2}$$

Note:

Since $e^{-j\pi n} = e^{+j\pi n}$ the answer can also be written

$$G(e^{j\omega}) = \frac{X(e^{j(\frac{\omega}{2}-\pi)}) + X(e^{j\omega/2})}{2}$$

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5.50 a) i)

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= \left(\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u[n-1]$$

$$= \left(\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{1}{2}$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{1}{2}$$

$$x[n] \xleftrightarrow{DTFT} \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{2} = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[n] \xleftrightarrow{DTFT} \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1-\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{3}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{1-\frac{1}{2}e^{-j\omega}}$$

$$h[n] = 3\left(\tfrac{1}{4}\right)^n u[n] - 2\left(\tfrac{1}{2}\right)^n u[n].$$

ii)

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1-\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{7}{12}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}}$$

$$Y(e^{j\omega})\left(1-\tfrac{7}{12}e^{-j\omega}+\tfrac{1}{12}e^{-j2\omega}\right) = X(e^{j\omega})\left(1-\tfrac{1}{2}e^{-j\omega}\right)$$

$$y[n] - \tfrac{7}{12}y[n-1] + \tfrac{1}{12}y[n-2] = x[n] - \tfrac{1}{2}x[n-1]$$