The joint probability mass function (jpmf) of two discrete r.v.s X and Y is

$$P_{X,Y}(x_i, y_j) = Pr(X = x_i, Y = y_j) \quad \{X = x_i\} \cap \{Y = y_j\}$$

Properties of jpmf:

1) $0 \le P_{X,Y}(x_i, y_j) \le 1$

2) $\sum_{x_i} \sum_{y_j} P_{X,Y}(x_i, y_j) = 1$

3) $Pr((X,Y) \in A) = \sum\sum_{(x_i, y_j) \in A} P_{X,Y}(x_i, y_j)$

4) $P_X(x_i) = \sum_{y_j} P_{X,Y}(x_i, y_j)$

$P_Y(y_j) = \sum_{x_i} P_{X,Y}(x_i, y_j)$

The joint cumulative distribution function (jcdf) of two r.v.s X and Y is

$$F_{X,Y}(x, y) = Pr(X \le x, Y \le y)$$

Properties of jcdf:

1) $0 \le F_{X,Y}(x, y) \le 1$

2) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

$F_{X,Y}(\infty, \infty) = 1$

3) $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$

if $x_1 \leq x_2$ <u>and</u> $y_1 \leq y_2$

4) $F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y)$

$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y)$

The <u>joint probability density function</u> (jpdf)

of two jointly continuous r.v.s X and Y is

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$

or equivalently

$$F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x', y')\, dx'\, dy'$$

<u>Properties of jpdf</u>:

1) $f_{X,Y}(x, y) \geq 0$

2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$

3) $\Pr((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$

4) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

The <u>mean</u> or <u>expected value</u> of $g(X,Y)$ is

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \sum_{y_j}\sum_{x_i} g(x_i,y_j)\, P_{X,Y}(x_i,y_j)$$

if X,Y discrete

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dx\,dy$$

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\,dx\,dy$$

Conditional cdf, pdf, pmf

Recall that if X is a r.v. and M is an event with $Pr(M) > 0$ then

$$F_X(x \mid M) = Pr(X \le x \mid M)$$

$$f_X(x \mid M) = \frac{d}{dx} F_X(x \mid M)$$

Let X and Y be jointly continuous r.v.s.

Let $M = \{Y = y\}$. Since $Pr(Y = y) = 0$,

$F_X(x \mid Y = y)$ and $f_X(x \mid Y = y)$

cannot be defined as above.

Instead the following definitions are used:

The conditional cdf of X given Y is

$$F_{X|Y}(x \mid y) = \lim_{dy \to 0} F_X(x \mid y < Y \le y + dy)$$

The conditional pdf of X given Y is

$$f_{X|Y}(x \mid y) = \frac{d}{dx} F_{X|Y}(x \mid y)$$

$$= \frac{f_{X,Y}(x, y)}{f_Y(y)}, \quad f_Y(y) > 0$$

Likewise,

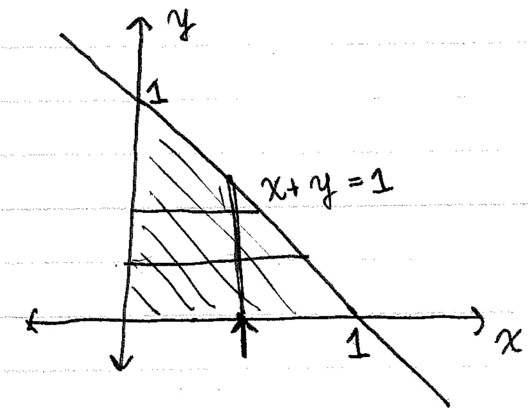
$$\cancel{f_X}\ f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad f_X(x) > 0$$

Ex: let X, Y be r.v.s with jpdf

$$f_{X,Y}(x,y) = \begin{cases} kx &, \quad 0 \le x+y \le 1,\ x>0,\ y>0 \\ 0 &, \quad \text{else} \end{cases}$$

k is a constant.

a) Find k



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \quad (=1)$$

$$= \int_0^1 \int_0^{1-y} kx\,dx\,dy$$

$$= k\int_0^1 \left.\frac{x^2}{2}\right|_0^{1-y} dy$$

$$= k\int_0^1 \frac{(1-y)^2}{2}\,dy \quad , \quad u = 1-y \ , \quad du = -dy$$

$$= k \int_0^1 \frac{u^2}{2}\, du = \frac{k}{6} = 1$$

$$\Rightarrow \boxed{k = 6}$$

b) Find $f_X(x)$, $f_Y(y)$, $f_{X|Y}(x|y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_0^{1-x} 6x\, dy \qquad , \quad 0 \le x \le 1$$

$$= 6xy \Big|_0^{1-x}$$

$$= 6x(1-x) \qquad , \quad 0 \le x \le 1$$

$$= 0 \qquad , \quad \text{else}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^{1-y} 6x\, dx \qquad , \quad 0 \le y \le 1$$

$$= 3(1-y)^2 \qquad , \quad 0 \le y \le 1$$

$$= 0 \qquad , \quad \text{else}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad 2 \quad 0 \le x+y \le 1,\ x,y>0$$

$$0 \le y \le 1$$

$$= \frac{2x}{(1-y)^2}, \quad \begin{matrix} 0 \le y \le 1 \\ 0 \le x \le 1-y \end{matrix}$$

$$= 0, \quad \text{else}$$