The Poisson Process

The poisson process consists of various R.V's which model events which occur at random points in continuous time. ($t \geq 0$).

The exponential R.V:

The PDF is given by

$$f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$
$$= 0 \quad , t < 0$$

where λ is a parameter with $\lambda > 0$.

Mean $\mu, \tilde{T}$: $\frac{1}{\lambda}$

Variance σ_T^2 : $\frac{1}{\lambda^2}$

The Erlang R.V: (of order k)

The PDF is

$$f_{T_k}(t) = \frac{\lambda(\lambda t)^{k-1} \cdot e^{-\lambda t}}{(k-1)!}, \quad t \geq 0$$
$$= 0 \quad , t < 0$$

where $\lambda > 0$ and $k = 1, 2, \ldots$ are the parameters.

Note: Erlang PDF of order 1 is an exponential.

The mean: $\mu = \dfrac{k}{\lambda}$

the variance: $\sigma_{T_k}^2 = \dfrac{k}{\lambda^2}$

The Erlang CDF is given by:

$$F_{T_k}(t) = \begin{cases} 1 - \sum_{n=0}^{k-1} \left\{ \dfrac{(\lambda t)^n e^{-\lambda t}}{n!} \right\}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The Poisson R.V :

The PMF is given by:

$$p_{K_t}(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, \ldots \quad t \geq 0$$

where $\lambda > 0$ and $t \geq 0$ are parameters.

The mean: $\mu_t = \lambda t$

The variance: $\sigma_{K_t}^2 = \lambda t$

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All of these R.V's come up in modelling the behavior of a sequence of events which occur at random points in continuous time. ($t \geq 0$)

These could be times that:

- The packets arrive at a node in a network.
- The particles that impinge on a detector.
- The customers arrive at a service station.
- Devices or components breakdown.

λ is the mean rate of occurence.

Specifically:

1. An exponential R.V is used to model the time up to first event and/or the time between the n^{th} and $n+1^{th}$ events for $n = 0, 1, \dots$
2. An Erlang R.V of order k is used to model the time up to the k^{th} event and/or the time between the n^{th} and $n+k^{th}$ events for $n = 0, 1, \dots$

3. The poisson R.V with parameter t is used to model the number of events which occur in the interval $[0, t]$ and/or in a interval $[s, s+t]$ for $s \geq 0$.

Example: Light bulbs with mean lifetime of 500 hours before burn out and are replaced immediately with a fresh bulb. Assume that the burn out times follow a poisson process.

1. Find the probability that the first bulb lasts at least 250 hours, 2. the conditional probability that the bulb lasts at least 500 hours given it lasts at least 250 hours, 3. and the conditional mean lifetime given it lasts atleast 250 hours.

Let T be the lifetime of 1^{st} bulb:

T is an exponential with $\bar{T} = \frac{1}{\lambda} = 500$

$$f_T(t) = \frac{1}{500} e^{-t/500}, \quad t \geq 0$$

$$= 0, \quad t < 0$$

(i) $P(T>250) = \int_{250}^{\infty} \frac{1}{500} e^{-t/500} dt = e^{-1/2}$

(ii) $P(T>500 / T>250) = \frac{P\{(T>500) \cap (T>250)\}}{P\{T>250\}}$

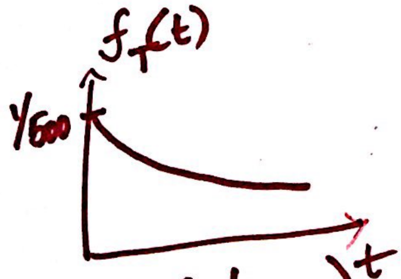
$$= \frac{P(T>500)}{P(T>250)}$$

$$= \frac{\int_{500}^{\infty} \frac{1}{500} e^{-t/500} dt}{e^{-1/2}}$$

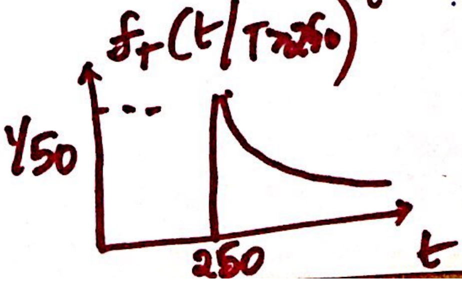
$$= e^{-1/2}$$

(iii) Now to get $E[T / T>250]$ need $f_T(t | T>250)$

$$f_T(t / T>250) = \frac{f_T(t)}{P(T>250)}, \quad t>250$$



$$= \frac{\frac{1}{500} e^{-t/500}}{e^{-1/2}}$$



$$= \begin{cases} \frac{1}{500} e^{-(t-250)/500}, & t>250 \\ 0, & \text{else} \end{cases}$$

$$E[T \mid T>250] = 250 + E[T]$$

$$= 750.$$