1. (Borel-Cantelli) Prove that if $\sum \mu(E_n) < \infty$ then
$$\mu\left(\bigcap_{j\geq 1}\bigcup_{n\geq j} E_n\right) = 0.$$

2. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is non-negative and satisfies the property that
$$\int_E f\ d\mu \leq \mu(E)^{1/q} \text{ for all measurable sets } E$$
(here $1 < q < \infty$ and p is such that $\frac{1}{p} + \frac{1}{q} = 1$). Prove that $f \in L^r(\mu)$ for $1 \leq r < p$.
Hint: Consider the sets $E_n = \{x \in X : 2^n \leq f(x) \leq 2^{n+1}\}$.

3. Let $f \in L^1(\mathbb{R}^n)$. Prove that the function
$$F(h) = \int_{\mathbb{R}^n} |f(x+h) - f(x)|dx$$
is continuous in h.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $1 < p \leq \infty$. Show $L^r(\mu) \supset L^p(\mu)$ whenever $1 \leq r < p$. Does the reverse containment hold?

5. Let $f(x) = e^{-|x|}, g \in L^1(\mathbb{R})$.
   (a) Show $||f * g||_1 \leq 2||g||_1$.
   (b) Show $f * g$ is differentiable on $\mathbb{R} - \{0\}$, and find $\frac{d}{dx}(f * g)$.

6. Let $(X, \Re, \mu)$ be a finite (or even σ -finite) measure space. Notice $L^\infty(\mu)$ is a ring with unity. Let I be a proper ideal of $L^\infty(\mu)$ with the following property:
$$\text{Whenever } f_n \in I, f \in L^\infty(\mu) \text{ such that } \forall g \in L^1(\mu), \textstyle\int f_n g \to \int fg, \text{ then} f \in I.$$
Show $\exists A \subset X$, with $\mu A > 0$ such that $I = \{\phi \in L^\infty : \forall x \in A, \phi(x) = 0\}$. **Hint:** If $\phi \in I$ show $\chi_{\{\phi \neq 0\}} \in I$.

7. For each $n \in \mathbb{N}$, let E_n be a Lebesgue measurable set and let μ be the Lebesgue measure.

   (a) Prove that
   $$\mu\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k\right) \leq \liminf_n \mu(E_n).$$

   (b) Exhibit a sequence $\{E_n\}$ for which equality in the above expression does not hold.

   (c) What can you say about the relation
   $$\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) \geq \limsup_n \mu(E_n)?$$

8. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^p(\mu), 1 \leq p < \infty$. Suppose E_n is a sequence of measurable sets satisfying $\mu(E_n) = \dfrac{1}{n}$ for all n. Prove that
   $$\lim_{n\to\infty}\left(n^{\frac{p-1}{p}}\int_{E_n}|f|d\mu\right) = 0.$$

9. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$ and let $F_1, \ldots, F_{17}$ be seventeen measurable subsets of X with $\mu(F_j) = \frac{1}{4}$, $\forall j$.

   (a) Prove $\exists$ five of our sets, say $F_{n_1}, ..., F_{n_5}$ with $\mu(\cap_{j=1}^{5} F_{n_j}) > 0$.

   (b) Is the conclusion above true if we take sixteen sets instead of seventeen?