MA544 Qual Review Problems 8 Bridges Summer 08

1. Let X be a normed linear space (for example, $L^{13.7}(\mathbb{R})$). Show X is complete iff every absolutely convergent series converges in X. (Let $x_n \in X$. We say $\sum x_n$ converges iff $\exists x \in X$ such that $\sum x_n = x$, and we say $\sum x_n$ converges absolutely iff $\sum ||x_n||$ is a convergent series of real numbers.

2. (a) Let $E = \{x \in [0,1] : x = .a_1a_2\cdots, a_n \neq 7\ \forall n\}$, i.e., the set E consists of the reals in $[0,1]$ whose decimal expansion do not contain a 7. Prove that $m(E) = 0$.

   (b) Let $E = \{x \in [0,1] : x = .a_1a_2\cdots\ ,\ \text{with } a_n = 2 \text{ or } 3\ \forall n\}$. Prove that E is Lebesgue measurable and compute its measure.

3. Let $f \in L^p[0,1], 2 \leq p \leq \infty$. Show $f = 0$ a.e if for all $x \in [0,1]$

$$\int_0^1 f(y)\sin(xy)dy = 0$$

   .

4. Let $(X, \mathcal{M}, \mu)$ be a probability space and $f_n : X \to [0,1]$ a sequence of measurable functions which converge to zero in measure. Let F be a uniformly continuous function on $\mathbb{R}$. Prove

$$\lim_n \int_X F(f_n) = F(0).$$

5. Let f be a measurable function on a measure space $(X, \mathcal{M}, \mu)$. For $0 < p < \infty$ define

$$\phi(p) = \int_X |f|^p d\mu.$$

   (a) Let $E = \{p|\phi(p) < \infty\}$. Show that if $r, s \in E$ and $r < p < s$ then $p \in E$.

   (b) Prove that $\log\phi$ is convex on the interval (r, s) whenever $r, s \in E$.

6. Let f be a measurable function on $[0,1]$. Find

$$\lim_{n\to\infty} \int_0^1 (\cos f(x))^{2n} dx$$

   .

7. Prove or disprove: Let $1 \leq p$. $\{f \in L^p[0,1] : ||f||_p \leq 1\}$ is compact.

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and $1 \leq p < q \leq \infty$. Let $A \subseteq L^q \cap L^p$ that is compact with respect to the L^q norm. Show A is compact in the L^p norm. Is the converse true?

9. Proof or counter-example: If f is a bounded measurable function on $(X, \mathcal{M}, \mu)$, then there exists a continuous function g with $g = f$, μ-a.e.