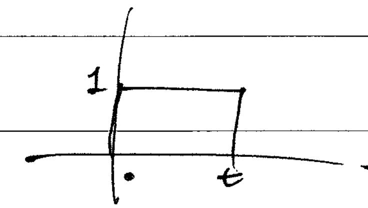
6/29 ①

Prob 2.22 (a)

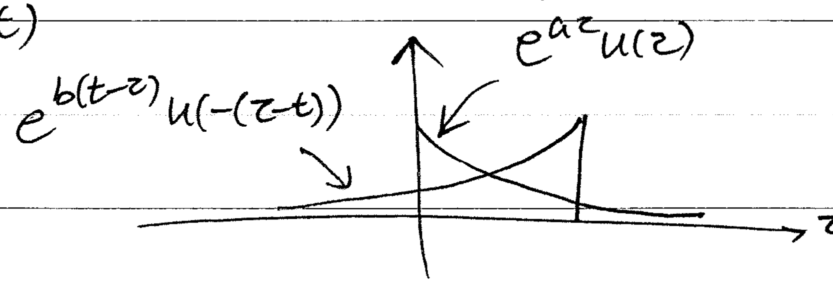
$$e^{at}u(t) * e^{bt}u(t) = ? \quad , \; b \neq a$$

$$y(t) = \int_{-\infty}^{\infty} e^{a\tau}u(\tau) \cdot e^{b(t-\tau)}u(t-\tau)\, d\tau$$

$$= e^{bt} \cdot \int_{-\infty}^{\infty} e^{(a-b)\tau} \underline{u(\tau)u(-(\tau-t))}\, d\tau$$



$$= e^{bt} \cdot \int_{0}^{t} e^{(a-b)\tau}\, d\tau \cdot u(t)$$



$$= e^{bt}\left\{ \frac{1}{a-b} e^{(a-b)\tau} \Big|_0^t \right\} \cdot u(t)$$

$$= \frac{e^{bt}}{a-b}\left\{ e^{(a-b)t} - e^{0} \right\} \cdot u(t)$$

$$= \frac{1}{a-b}\left\{ e^{at} - e^{bt} \right\} u(t)$$

- For CT LTI System with impulse response $h(t)$, if the inverse system exists, it must satisfy:

$$h(t) * h_I(t) = \delta(t)$$

where $h_I(t)$ is the impulse response of the inverse system.

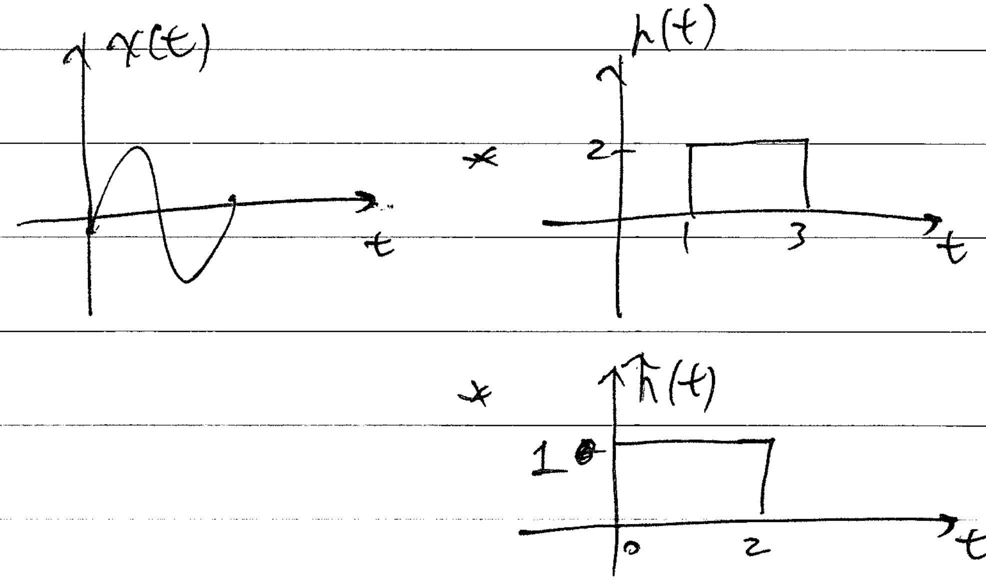
- Although it's peculiar to talk about the impulse response of a differentiator, one can use Leibniz's Integral Rule to show the following inverse system pair.

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t} x(\tau)\,d\tau} \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow x(t)$$

Since the order of LTI system in series doesn't matter, we also have

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \xrightarrow{y(t)} \boxed{\int_{-\infty}^{t} y(\tau)\,d\tau} \longrightarrow x(t)$$

Prob 2.22 (c) — Do it Yourself



$= y(t)$

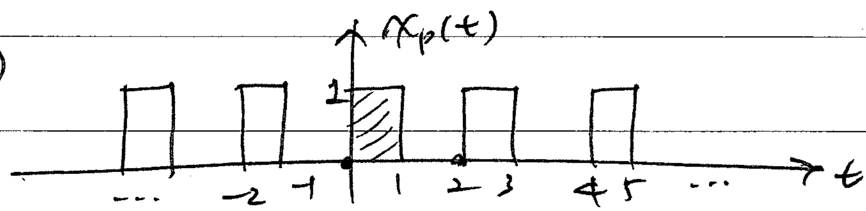
$= \hat{y}(t)$

$$\underline{y(t) = 2\,\hat{y}(t-1)}$$

- Any periodic signal with period T can be expressed as

$$x_p(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$$

e.g.)



$$x_p(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}\left(t-\tfrac{1}{2}-k\cdot 2\right) \quad , \quad \mathrm{rect}\left(t-\tfrac{1}{2}\right) = u(t)-u(t-1)$$

"Consider periodic signal $x_p(t)$ as input to LTI system with impulse response $h(t)$.

$$x_p(t) \longrightarrow \boxed{h(t)} \longrightarrow y_p(t)$$

$y_p(t)$ will also be periodic with period T."

→ simple to show using superposition / distributive property of convolution and time-invariance.

$$y_p(t) = x_p(t) * h(t) = \left\{ \sum_{k=-\infty}^{\infty} x(t-kT) \right\} * h(t)$$

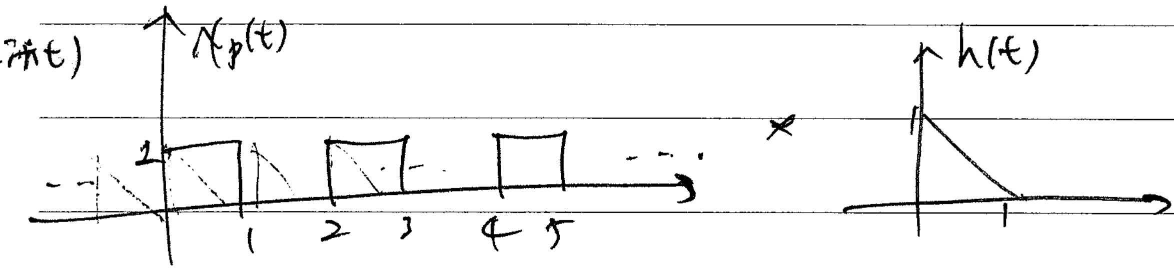
$$= \sum_{k=-\infty}^{\infty} \underbrace{x(t-kT) * h(t)}_{= y(t-kT)}$$

$$= \sum_{k=-\infty}^{\infty} y(t-kT), \quad \text{where } y(t) = x(t) * h(t)$$

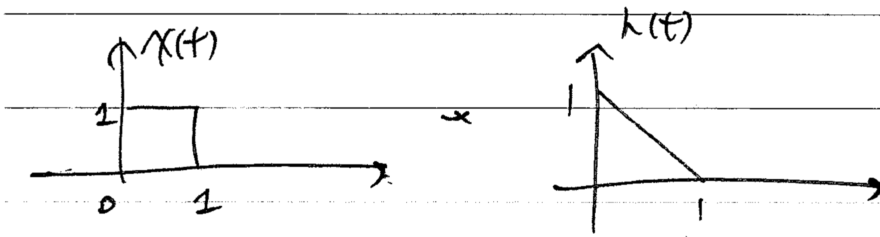
→ So, to find the output $y_p(t)$, you only have to convolve one period of $x_p(t)$ with $h(t)$ to form $y(t) = x(t) * h(t)$ and then repeat $y(t)$ every T sec.

Prob 2.22 (e) — Do-It-Yourself

(hint)



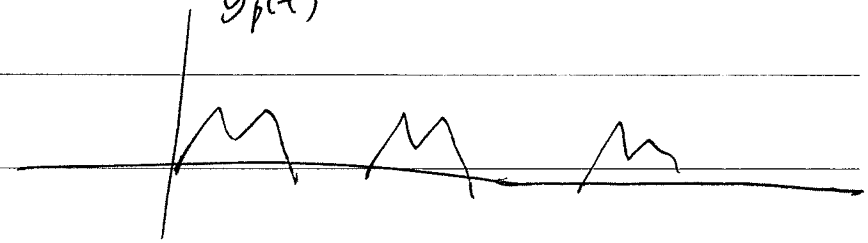
$= y_p(t)$?



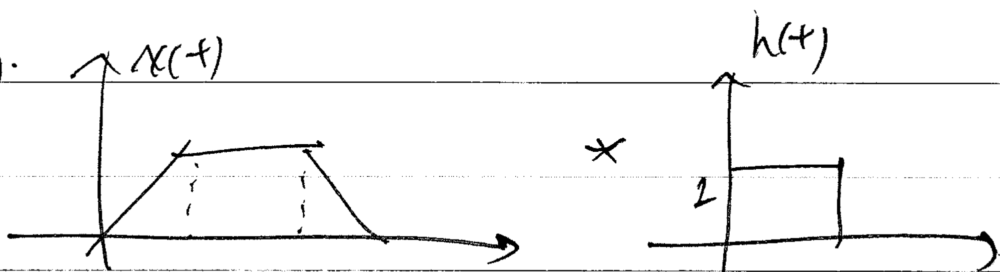
$= y(t)$

$$x_p(t) = \sum_{k=-\infty}^{\infty} x(t - k \cdot 2) \text{ period}$$

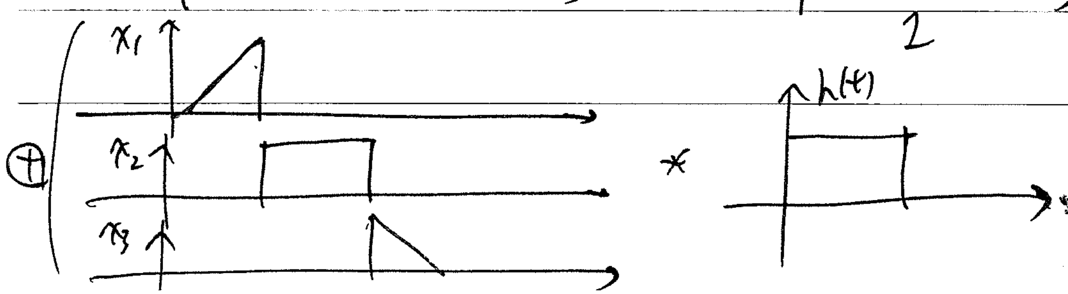
$$\rightarrow y_p(t) = \sum_{k=-\infty}^{\infty} y(t - k \cdot 2)$$



e.g.



$= y(t)$?



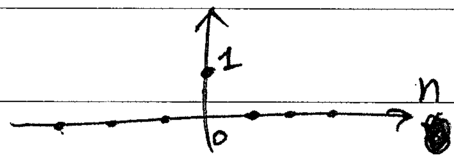
$\Rightarrow$ y_1, y_2, y_3 ⊕

3. Discrete-Time LTI system

1) Define impulse response of DT system that is both Linear and Time-invariant (LTI)

$\delta[n] \longrightarrow \boxed{LTI} \longrightarrow h[n]$: "Impulse Response"

<Input> <output>



To easily derive DT conv. formula, we view $x[n]$ (Input) as a sum of amplitude-scaled and time-shifted delta functions. (See Fig 2.1 p.976)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

- Time-Invariance dictates:

$\delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow h[n-k]$

- Homogeneity aspect of linearity dictates:

$x[k]\,\delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow x[k]\,h[n-k]$

- superposition aspect of linearity dictates:

$$\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= x[n] * h[n]$$

"DT convolution"