ECE 301 Signals and Systems
Homework # 4 Solution

8. O+W 2.55 — Impulse response from difference equation.

(a) Syst. initially at rest described by

$$y[n] - \tfrac{1}{2} y[n-1] = x[n]$$

Suppose $x[n] = \delta[n]$. Find $y[0]$.

Initially at rest means $y[-1] = 0$. Thus

$$y[0] = \tfrac{1}{2} y[-1] + \delta[0] = 1.$$

For $n \geq 1$ h satisfies

$$h[n] - \tfrac{1}{2} h[n-1] = 0$$

The correct initial condition is $h[0]=1$. The char. eqN is $r - \frac{1}{2} = 0 \implies$ root $= 1/2$

$$\therefore\ h[n] = c\left(\tfrac{1}{2}\right)^n \quad n \geq 0 \quad \text{with } h[0]=1$$

$$\implies c = 1$$

$$\therefore\ h[n] = \left(\tfrac{1}{2}\right)^n u[n].$$

(b) LTI system initially at rest is

$$y[n] - \tfrac{1}{2} y[n-1] = x[n] + 2x[n-1]$$

In block diagram form have two equivalent realizations

$$x[n] \longrightarrow \boxed{\overset{\#1}{z[n] = x[n] + 2x[n-1]}} \longrightarrow \boxed{\overset{\#2}{y[n] - \tfrac{1}{2} y[n-1] = z[n]}} \longrightarrow y[n]$$

Clearly impulse resp. of #1 is $h_1[n] = \delta[n] + 2\delta[n-1]$ and impulse resp. of #2 is same as in (a)

$$h_2[n] = \left(\tfrac{1}{2}\right)^n u[n].$$

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Then either by looking at the cascade with $\to \boxed{\#2} \to \boxed{\#1} \to$ or by convolving the two impulse responses we find

$$
\begin{aligned}
h_b[n] &= \text{impulse response for (b)} \\
&= h_a[n] + 2h_a[n-1] \qquad \left(\begin{matrix} h_a = \text{imp. resp.} \\ \text{of (a)} \end{matrix} \right). \\
&= \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \underbrace{\left(\tfrac{1}{2}\right)^n + 2\left(\tfrac{1}{2}\right)^{n-1}}_{5\left(\frac{1}{2}\right)^n} & n \geq 1 \end{cases}
\end{aligned}
$$

(c) With $h_a[n] = \left(\frac{1}{2}\right)^n u[n]$ the output in resp. to an input $x[n]$ can be computed as

$$y[n] = h_a * x[n] = \sum_k h_a[n-k]\, x[k]$$

Plugging this into the difference equation we want to show

$$\sum_{k=-\infty}^{\infty} h_a[n-k]\, x[k] - \frac{1}{2} \sum_{k=-\infty}^{\infty} h_a[n-1-k]\, x[k] = x[n]$$

$$\sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{n-k} x[k] - \frac{1}{2} \sum_{\ell=-\infty}^{n-1} \left(\tfrac{1}{2}\right)^{n-1-\ell} x[\ell]$$

$$= \sum_{k=-\infty}^{n-1} \underbrace{\left\{ \left(\tfrac{1}{2}\right)^{n-k} - \left(\tfrac{1}{2}\right)^{n-k} \right\}}_{0} x[k] + x[n]$$

$$= x[n].$$

(d) $$\sum_{k=0}^{N} a_k y[n-k] = x[n] \qquad a_0 \neq 0 \qquad (*)$$

If $x[n] = \delta[n]$ and system is at rest (ie $y[k]=0$ for $k<0$) before this happens then evaluating the diff. eqn for $n=0$

$$a_0 y[0] + \underbrace{\sum_{k=1}^{N} a_k y[-k]}_{0} = \delta[0] = 1$$

$$\therefore\ y[0] = 1/a_0$$

Thus impulse resp. solves $\sum_{k=0}^{N} a_k h[n-k] = 0$ for $n>0$ with $h[k]=0$ for $k<0$ and $h[0] = 1/a_0$.

Following procedure used in (b) if $h[n]$ is impulse response of system $(*)$, then

$$\tilde{h}[n] \triangleq \sum_{k=0}^{M} b_k h[n-k]$$

is impulse response for

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (**).$$

$a_0 \neq 0$

(e) Direct solution of (**) for $x[n] = \delta[n]$ and at rest before time 0 ie $y[k]=0$ for $k<0$.

The equation becomes

$$\sum_{k=0}^{N} a_k h[n-k] = \sum_{k=0}^{M} b_k \delta[n-k] \qquad (***)$$

where $h[k]=0$ for $k<0$. For $n>M$ the right hand side is zero and

$$\sum_{k=0}^{N} a_k h[n-k] = 0 \quad \text{for } n > M$$

Thus $h[n]$ will have a general form the same as the solution to the homog. difference equation corresponding to (**). Only need figure out enough initial conditions. The initial conditions can be obtained by recursively solving (***) e.g.

$$\begin{aligned}
n&=0 & a_0 h[0] &= b_0 \\
n&=1 & a_0 h[1] + a_1 h[0] &= b_1 \\
n&=2 & a_0 h[2] + a_1 h[1] + a_2 h[0] &= b_2 \\
&\vdots \\
n&=M & a_0 h[M] + a_1 h[M-1] + \dots + a_M h[0] &= b_M
\end{aligned}$$

The above triangular system of equations can be solved for $h[0] \dots h[M]$.

$$\begin{aligned}
n&=M+1 & a_0 h[M+1] + a_1 h[M] + \dots + a_M h[1] &= 0 \\
&\vdots \\
n&=M+N & a_0 h[M+N] + a_1 h[M+N-1] + \dots + a_M h[N] &= 0
\end{aligned}$$

Solve these for $h[M+1] \dots h[M+N]$ and have initial conditions needed to specify constants in general soln.

(f-i) $y[n] - y[n-2] = x[n]$

Solve homog. eqn. $h[n] - h[n-2] = 0$ $n>0$ with $h[0] = 1$.

Char. Eqn. $r^2 - 1 = 0$ Roots $r = 1, -1$

$$\therefore \; h[n] = c_1 (1)^n + c_2 (-1)^n \quad n>0$$

$$h[1] - h[-1] = 0 \implies h[1] = 0$$
$$h[2] - h[0] = 0 \implies h[2] = 1$$

Solve for coeffs

$$\left.\begin{array}{l} 0 = c_1 - c_2 \\ 1 = c_1 + c_2 \end{array}\right. \implies c_1 = c_2 = 1/2$$

$$\therefore \; h[n] = \tfrac{1}{2}\left(1 + (-1)^n\right) \quad n>0$$
$$= \begin{cases} 1 & n \text{ even}, >0 \\ 0 & n \text{ odd}, >0 \\ 1 & n = 0 \\ 0 & n < 0. \end{cases}$$

(f-ii). $y[n] - y[n-2] = x[n] + 2x[n-1]$

Let $h_i[n] = \frac{1}{2}\left(1 + (-1)^n\right) u[n]$ denote impulse response from part (i). Using method of (b) and (d) we have

$$h_{ii}[n] = h_i[n] + 2h_i[n-1]$$
$$= \tfrac{1}{2}\left(1 + (-1)^n\right) u[n] + \left(1 + (-1)^{n-1}\right) u[n-1]$$

(f-ii) continued

Clearly for $n<0$ $h_{ii}[n]=0$. For $n=0$ $h_{ii}[0]=1$,
For $n \geq 1$

$$h_{ii}[n] = \frac{1}{2}\left(1+(-1)^n\right) + 1 + (-1)^{n-1}$$
$$= \frac{3}{2} + \frac{1}{2}(-1)^n - (-1)^n$$
$$= \frac{3}{2} - \frac{1}{2}(-1)^n = \begin{cases} 1 & n \text{ odd}, n \geq 1 \\ 2 & n \text{ even}, n \geq 1. \end{cases}$$

(f-iii)

Same idea as in previous part.

$$h_{iii}[n] = 2h_i[n] - 3h_i[n-4]$$

For $n<0$ $h_{iii}[n] = 0$

$n=0$ $h_{iii}[0] = 2h_i[0] = 2$

$n=1$ $h_{iii}[1] = 2h_i[1] = 0$

$n=2$ $h_{iii}[2] = 2h_i[2] = 2$

$n=3$ $h_{iii}[3] = 2h_i[3] = 0$

For $n \geq 4$

$$h_{iii}[n] = 2\cdot\frac{1}{2}\left(1+(-1)^n\right) - 3\frac{1}{2}\left(1+(-1)^{n-4}\right)$$
$$= 1 + (-1)^n - \frac{3}{2} - \frac{3}{2}(-1)^{-4}(-1)^n$$
$$= -\frac{1}{2} - \frac{1}{2}(-1)^n$$
$$= -\frac{1}{2}\left(1+(-1)^n\right) = \begin{cases} -1 & n \text{ even}, n \geq 4 \\ 0 & n \text{ odd}, n \geq 4. \end{cases}$$

(f-iv) $y[n] - \left(\frac{\sqrt{3}}{2}\right) y[n-1] + \frac{1}{4} y[n-2] = x[n]$

$\therefore$ $h[n]$ solves

$$h[n] - \left(\frac{\sqrt{3}}{2}\right) h[n-1] + \frac{1}{4} h[n-2] = 0 \quad n > 0$$

with $h[k] = 0$ $k < 0$, $h[0] = 1$.

Char. eqn.

$$r^2 - \left(\frac{\sqrt{3}}{2}\right) r + \frac{1}{4} = 0$$

Roots:

$$r = \frac{\sqrt{3}/2 \pm \sqrt{\frac{3}{4} - 1}}{2} = \frac{\sqrt{3}}{4} \pm \frac{j}{4}$$
$$= \frac{1}{2} e^{\pm j \pi/6}$$

$\therefore$ general form of solution is

$$h[n] = c_1 \left(\frac{1}{2}\right)^n \cos\left(\frac{n\pi}{6}\right) + c_2 \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{6}\right)$$
$$= \left(\frac{1}{2}\right)^n \left[c_1 \cos\left(\frac{n\pi}{6}\right) + c_2 \sin\left(\frac{n\pi}{6}\right)\right] \quad n \geq 1.$$

$$h[1] = \frac{\sqrt{3}}{2}$$
$$h[2] = \left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

Solving for coeffs gives.

$$h[n] = \left(\frac{1}{2}\right)^n \left[\cos\left(\frac{n\pi}{6}\right) + \sqrt{3} \sin\left(\frac{n\pi}{6}\right)\right] u[n].$$

9. O + W various parts.

This problem is closely related to the class lecture on "Finding impulse responce from differential equation." I actually like this treatment better than the way we did it in class. Here we rely on properties of Dirac deltas and this seems to simplify and shorten the argument. Start from (b).

(b) LTI initially at rest descibed by differential eqn

$$(*) \quad \sum_{k=0}^{N} a_k y^{(k)}(t) = x(t) \qquad a_N \neq 0,\ x(\cdot) = \text{input},\ y(\cdot) = \text{output}$$

with initial conditions (for application of signal at $t=0$) of

$$y(0^-) = 0,\ y^{(1)}(0^-) = 0 \ldots\ y^{(N-1)}(0^-) = 0.$$

Let $x(t) = \delta(t)$ and integrate both sides of differential eqn from $0-$ to $0+$

$$\sum_{k=0}^{N} a_k \int_{0^-}^{0^+} y^{(k)}(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

Since indefinite integral of $y^{(k)}(t)$ is $y^{(k-1)}(t)$ have

$$1 = \sum_{k=1}^{N} a_k y^{(k-1)}(t)\Big|_{t=0^-}^{0^+} + a_0 \int_{0^-}^{0^+} y(t)\, dt$$

$$= a_N\left[y^{(N-1)}(0^+) - \cancel{y^{(N-1)}(0^-)}^{\,0}\right] + a_{N-1}\left[y^{(N-2)}(0^+) - \cancel{y^{(N-2)}(0^-)}^{\,0}\right] + \cdots$$

$$\cdots + a_1\left[y(0^+) - \cancel{y(0^-)}_{\,0}\right] + a_0 \int_{0^-}^{0^+} y(t)\, dt$$

$$= a_N y^{(N-1)}(0^+) + a_{N-1} y^{(N-2)}(0^+) + \cdots + a_1 y(0^+) + a_0 \int_{0^-}^{0^+} y(t)\, dt$$

Now we must argue that $y^{(N-2)}(0^+) = 0 \ldots\ y(0^+) = 0$ and $a_0 \int_{0^-}^{0^+} y(t)\, dt = 0$. This is where we need to toss out mathematical rigor and argue physically/quasi-mathematically.

Smoothness of a function $y(t)$ refers to how many times we may differentiate it and still get a continuous function. Everytime we differentiate we get less smooth. Consider

$$y(t) \quad y^{(1)}(t) \quad y^{(2)}(t) \quad \cdots \quad y^{(N-1)}(t) \quad y^{(N)}(t)$$

$$\longrightarrow$$

smoothness decreases as we take more derivatives, ie the functions are more likely to be discontinuous ie have jumps or even deltas at $t=0$.

But must also note that there must be a certain number of continuous derivatives at $t=0$ in order that derivatives even be defined.

Conclude that the solution $y(t)$ to ⊛ with $x(t)=\delta(t)$ will have:

$y^{(N)}(t)$ has delta at $t=0$ to "match" delta on right

$y^{(N-1)}(t)$ has jump discontinuity (ie step) at $t=0$

$y^{(N-2)}(t) \;\cdots\; y^{(1)}(t)$, $y(t)$ are all continuous at $t=0$

$$\therefore \quad y^{(N-2)}(0^+) = y^{(N-2)}(0^-) = 0$$

$$\vdots$$

$$y^{(1)}(0^+) = y^{(1)}(0^-) = 0$$

$$y(0^+) = y(0^-) = 0$$

$$\int_{0^-}^{0^+} y(t)\, dt = 0$$

$\therefore$ must have $1 = a_N\, y^{(N-1)}(0+)$

$$y^{(N-1)}(0^+) = 1/a_N.$$

This gives the initial conditions that the impulse response must satisfy for diff. eqn. of form (*).

(c) Properties of LTI systems imply that if $h(t)$ is soln to

$$\sum_{k=0}^{N} a_k h^{(k)}(t) = 0 \quad \text{for } t>0 \quad \text{with } h(0^+)=0,\ h^{(1)}(0^+)=0, \ldots\ h^{(N-2)}(0^+)=0,\ h^{(N-1)}(0^+)=1/a_N$$

Then the impulse response of the system

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = \sum_{k=0}^{M} b_k x^{(k)}(t) \qquad (\text{at rest at } t=0^-)$$

is

$$h_{overall}(t) = \sum_{k=0}^{M} b_k h^{(k)}(t) \qquad \text{for } t \geq 0. \qquad (**)$$

(e) Lets do this part now while we are in theory mode.
Let $h(t)$ be solution to

$$\sum_{k=0}^{N} a_k h^{(k)}(t) = \delta(t) \qquad t \geq 0 \quad h(0^-)=0,\ h^{(1)}(0^-)=0 \ldots h^{(N-1)}(0^-)=0.$$

As we argued (non rigorously) in part (b):

$h^{(N)}(t)$ has delta at $t=0$

$h^{(N-1)}(t)$ has step discont. at $t=0$

$h^{(N-2)}(t) \ldots h(t)$ are continuous at $t=0$.

Thus if in (**)

$M < N \implies h_{overall}(t)$ will have no deltas

$M = N \implies$ " " " a delta

$M > N \implies$ " " " a delta, and derivatives of delta up to order $M-N$.

(d-i) $y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = x(t)$.

To find $h(t)$ $t \geq 0$ solve homog. equation with initial conditions

$$h(0^+) = 0, \quad h^{(1)}(0^+) = 1.$$

Char. eqn. is $s^2 + 3s + 2 = 0$

$$(s+1)(s+2) = 0 \Rightarrow s = -1, -2$$

$$\therefore \ h(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad t \geq 0.$$

Solve for constants

$$h(0^+) = 0 = c_1 + c_2 \Longrightarrow c_1 = -c_2$$

$$h^{(1)}(0^+) = 1 = -c_1 - 2c_2$$

$$1 = c_2 - 2c_2 = -c_2$$

$$\therefore \ c_2 = -1, \ c_1 = 1.$$

$$h(t) = \left(e^{-t} - e^{-2t}\right) u(t).$$

(f-ii) $y^{(2)}(t) + 5y^{(1)}(t) + 6y(t) = x^{(3)}(t) + 2x^{(2)}(t) + 4x^{(1)}(t) + 3x(t)$.

First solve $h^{(2)}(t) + 5h^{(1)}(t) + 6h(t) = 0$ for $t \geq 0$ with $h(0^+) = 0$, $h^{(1)}(0^+) = 1$. The char. eqn. is

$$s^2 + 5s + 6 = 0$$

$$s = \frac{-5 \pm \sqrt{25-24}}{2} = -\frac{5}{2} \pm \frac{1}{2}$$

$$= -3, -2$$

$$\therefore \ h(t) = c_1 e^{-2t} + c_2 e^{-3t} \qquad t \geq 0.$$

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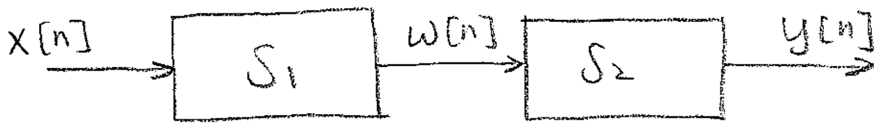
$$y[n] = -ay[n] + b_0 x[n] + b_1 x[n-1] \quad \text{——— } ①$$

a)

$$S_1: \; y_1[n] = b_0 x_1[n] + b_1 x_1[n-1]$$

$$S_2: \; y_2[n] = -a y_2[n-1] + x_2[n]$$

i) If we put S_1 before S_2.



then

$$w[n] = b_0 x[n] + b_1 x[n-1] \quad \text{——— } ②$$

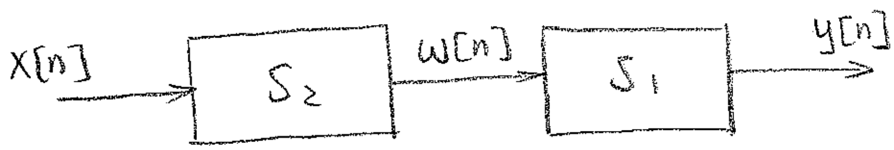
$$y[n] = -a y[n-1] + w[n] \quad \text{——— } ③$$

Plugging ② → ③

$$y[n] = -ay[n-1] + b_0 x[n] + b_1 x[n-1],$$

which is exactly equation ①.

ii) If we put S_2 before S_1.



then

$$w[n] = -a w[n-1] + x[n] \quad \text{——— } ④$$

$$y[n] = b_0 w[n] + b_1 w[n-1] \quad \text{——— } ⑤$$

From ④, $w[n-1] = -a w[n-2] + x[n-1]$.

From ⑤, $y[n-1] = b_0 w[n-1] + b_1 w[n-2]$

Plugging ⑤ into the right hand side of ①.

$$-a\,y[n-1] + b_0 x[n] + b_1 x[n-1]$$

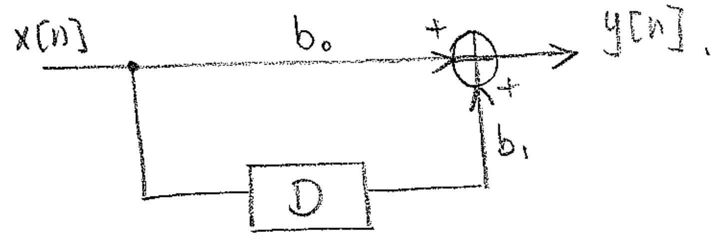
$$= -a b_0 w[n-1] - a b_1 w[n-2] + b_0 x[n] + b_1 x[n-1]$$

$$= b_0\{-a w[n-1] + x[n]\} + b_1\{-a w[n-2] + x[n-1]\}$$

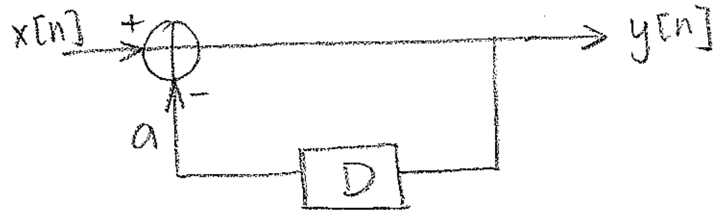
$$= b_0 w[n] + b_1 w[n-1]$$

$$= y[n].$$

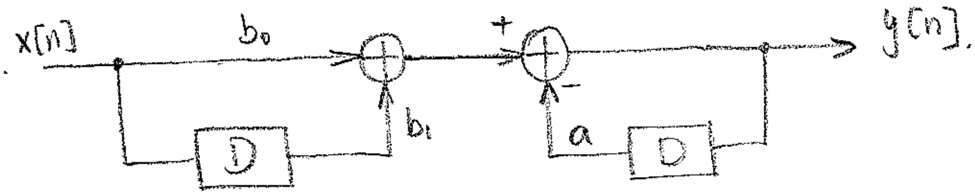
b) S_1 :



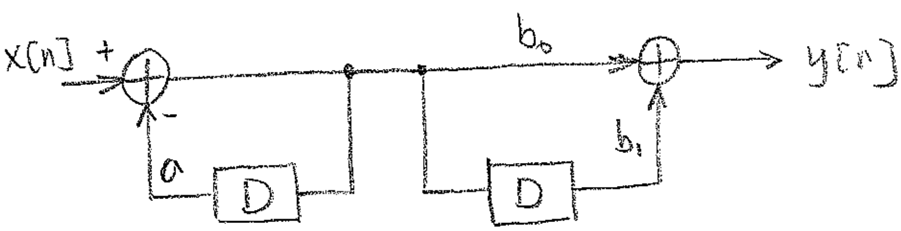
c) S_2 :



d)



e)



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