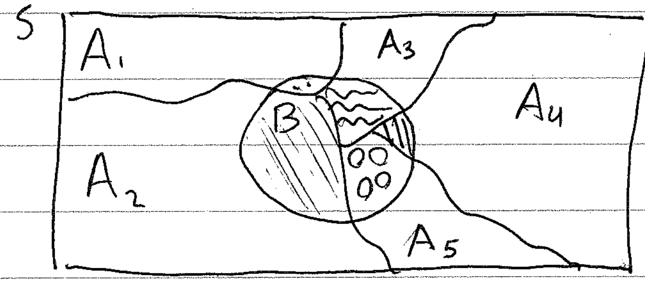
Total Probability Theorem:

Let B be an event, and $A_1, \ldots, A_n$ be a collection of mutually exclusive and collectively exhaustive events.



Since $S = A_1 \cup A_2 \cup \ldots \cup A_n$ we can express B as follows:

$$
\begin{aligned}
B &= B \cap S \\
&= B \cap (A_1 \cup \ldots \cup A_n) \\
&= (B \cap A_1) \cup \ldots \cup (B \cap A_n)
\end{aligned}
$$

and since $(B \cap A_1) \cap \ldots \cap (B \cap A_n) = B \cap (A_1 \cap \ldots \cap A_n)$
$= \emptyset$

we have that

$$Pr(B) = Pr(B \cap A_1) + \ldots + Pr(B \cap A_n)$$, by axiom (iii)

$$Pr(B) = Pr(B \cap A_1) \cup \ldots \cup (B \cap A_n)$$

But $Pr(B \cap A_i) = Pr(B \mid A_i)\, Pr(A_i)$

$$\Rightarrow \boxed{\begin{aligned} Pr(B) &= Pr(B \mid A_1) Pr(A_1) + \ldots + Pr(B \mid A_n) Pr(A_n) \\ &= \sum_{i=1}^{n} Pr(B \mid A_i) Pr(A_i) \end{aligned}}$$

Total Probability Theorem

We can use the above expression for $Pr(B)$ to obtain a different formulation of Bayes Rule

$$Pr(A_j \mid B) = \frac{Pr(B \mid A_j)\, Pr(A_j)}{Pr(B)}$$

$$= \frac{Pr(B \mid A_j)\, Pr(A_j)}{\sum_{i=1}^{n} Pr(B \mid A_i)\, Pr(A_i)}$$

Ex An urn contains 3 red balls and 5 green balls. Pick two balls without replacement and note the sequence of colors.

Let A be the event that the first ball is red.
Let B be the event that the second ball is red.

a) Find $Pr(A \cap B)$

$Pr(A \cap B) = Pr(B|A)\,Pr(A)$

$Pr(A) = 3/8$

$Pr(B|A) = 2/7$

$Pr(A \cap B) = 3/8 \cdot 2/7 = 3/28$

b) Find $Pr(B)$

Use Total Probability Theorem

$A \cup \bar{A} = S$, $A \cap \bar{A} = \emptyset$

$A, \bar{A}$ forms a partition of S

$Pr(B) = Pr(B|A)\,Pr(A) + Pr(B|\bar{A})\,Pr(\bar{A})$

$Pr(\bar{A}) = 5/8$

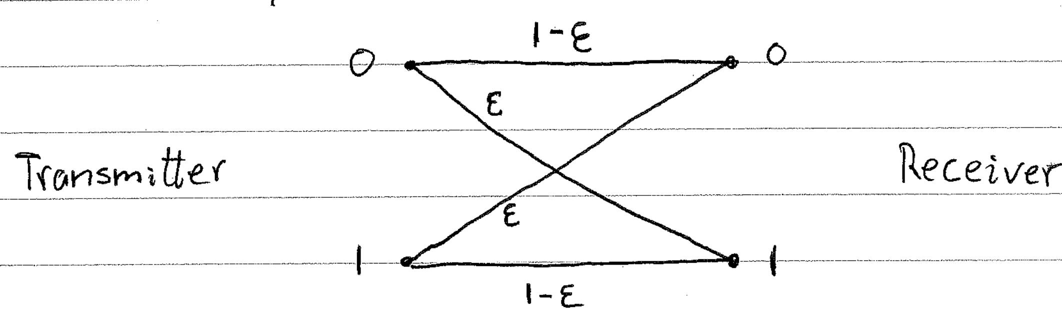
$Pr(B|\bar{A}) = 3/7$

$Pr(B) = 3/8 \cdot 2/7 + 5/8 \cdot 3/7$

$= 3/8$

Ex Binary Communication System

In digital communications systems, data is sent as a sequence of bits (0 or 1). The simplest model for such a system is the binary symmetric channel.



Channel

The transmitter sends a 0 or 1 ~~aerfe~~ across the channel using a corresponding signal. The receiver decodes what was sent based on the signal it receives. The signal may be distorted by noise in the channel, so random errors are made by the receiver.

Let T_i be the event that i is sent $(i = 0, 1)$
Let R_i be the event that i is received $(i = 0, 1)$
The channel model above defines the following transition probabilities

$$Pr(R_0 | T_0) = Pr(R_1 | T_1) = 1 - \varepsilon = 0.9$$
$$Pr(R_0 | T_1) = Pr(R_1 | T_0) = \varepsilon = 0.1$$

Given that $Pr(T_0) = p = 0.6$, $Pr(T_1) = 1-p = 0.4$
want to find the following:

a) $Pr(R_0)$ and $Pr(R_1)$

Use ~~Tota~~ Total Probability Theorem

$$Pr(R_0) = Pr(R_0 | T_0) Pr(T_0) + Pr(R_0 | T_1) Pr(T_1)$$
$$= (1-\varepsilon) p + \varepsilon (1-p)$$
$$= 0.9 \cdot 0.6 + 0.1 \cdot 0.4 = 0.58$$

$$Pr(R_1) = 1 - Pr(R_0)$$
$$= \varepsilon p + (1-\varepsilon)(1-p)$$
$$= 0.42$$

b) $Pr(T_0 | R_0)$ and $Pr(T_1 | R_1)$

Bayes Rule

$$Pr(T_0 | R_0) = \frac{Pr(R_0 | T_0)\, Pr(T_0)}{Pr(R_0)}$$
$$= \frac{0.9 \cdot 0.6}{0.58}$$
$$= 0.9310$$

Similarly $Pr(T_1 | R_1) = 0.8571$

c) $\Pr(\text{error})$

Event "error" $= (T_0 \cap R_1) \cup (T_1 \cap R_0)$

$$\Pr(\text{error}) = \Pr(T_0 \cap R_1) + \Pr(T_1 \cap R_0)$$
$$= \Pr(R_1 | T_0)\Pr(T_0) + \Pr(R_0 | T_1)\Pr(T_1)$$
$$= 0.1 \cdot 0.6 + 0.1 \cdot 0.4$$
$$= 0.1 = \varepsilon$$