- Multiplication (in time) Property

$$z[n] = x[n]\,y[n] \overset{DTFT}{\longleftrightarrow} Z(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)\,Y(\omega-\mu)\,d\mu$$

$$= \frac{1}{2\pi} X(\omega) \circledast Y(\omega)$$

periodic convolution.

- limits on integral are $-\pi$ to π (not $-\infty$ to ∞)
- $X(\omega)$ and $Y(\omega)$ are both periodic with period 2π. Must factor in when forming $Y(\omega-\mu) = Y(-(\mu-\omega))$

- $$n\,x[n] \overset{DTFT}{\longleftrightarrow} j\frac{dX(\omega)}{d\omega}$$

discrete. continuous.

cannot take derivative wrt n in DT, but can take derivative wrt ω in freq. domain.

note if $x[n] = x_a(nT_s)$, and $T_s \ll 1$

$$\frac{1}{T_s}\left(x_a((n+1)T_s) - x_a(nT_s)\right) = \frac{1}{T_s}\left(x[n+1] - x[n]\right)$$

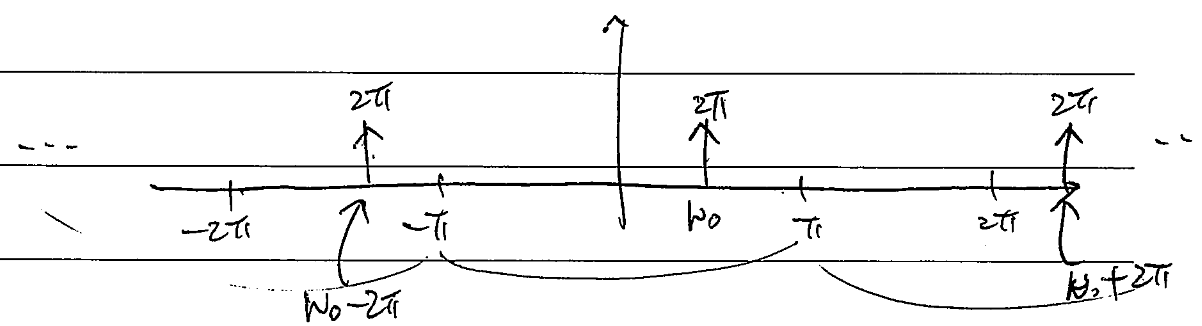
is a good approx. to the derivative of $x_a(t)$ at the time $t = nT_s$.

Thus, might consider this property

$$\frac{1}{T_s}\left(x[n+1] - x[n]\right) \overset{DTFT}{\longleftrightarrow} \frac{1}{T_s}\left(e^{j\omega} - 1\right) X(\omega)$$

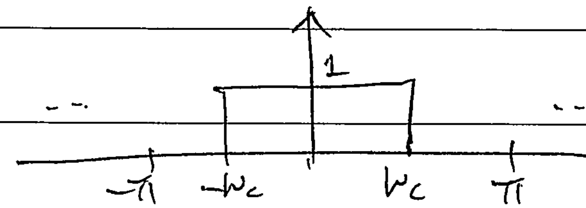
- See Table 5.1 for DTFT properties.
- DTFT pairs

$$x[n] = e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - k2\pi)$$



* We will often plot only over $-\pi < \omega < \pi$ but must keep in mind that $X(\omega)$ is periodic (2π).

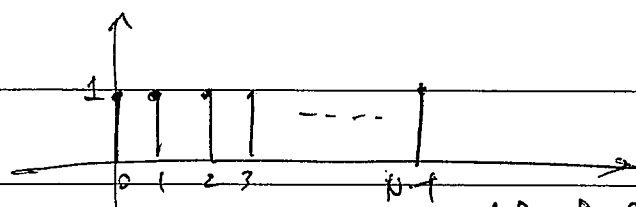
- $$x[n] = \frac{\sin(\omega_c n)}{\pi n} \xleftrightarrow{\text{DTFT}}$$



$(-\pi < \omega_c < \pi)$

- DT rectangle

$$x[n] = u[n] - u[n-N] \xleftrightarrow{\text{DTFT}} X(\omega) = \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{N-1}{2}\omega}$$



(Proof)

$$X(\omega) \overset{\text{def. of DTFT}}{=} \sum_{n=0}^{N-1} (1)\, e^{-j\omega n} = \sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

- use "half-angle trick" to simplify

$$X(\omega) = \frac{e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \cdot \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\omega\frac{1}{2}}} \cdot \frac{\frac{1}{2j}}{\frac{1}{2j}}$$

$$= \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{N-1}{2}\omega}$$

almost in polar form, but $\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$ ~~are~~ can go negative for certain freq bands.

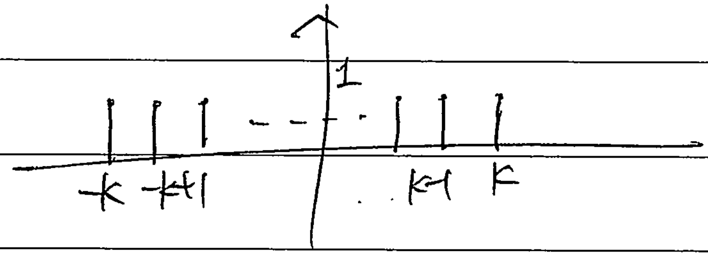
- Suppose N is odd such that $N = 2k+1$ and $k = \frac{N-1}{2}$ is an integer

- Form $y[n] = x[n+k] \Rightarrow$ shift to left by k so that DT rectangle is centered at $n=0$

$$Y(\omega) = e^{jk\omega} X(\omega) = \cancel{e^{j\frac{N-1}{2}\omega}}\, \cancel{e^{-j\frac{N-1}{2}\omega}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

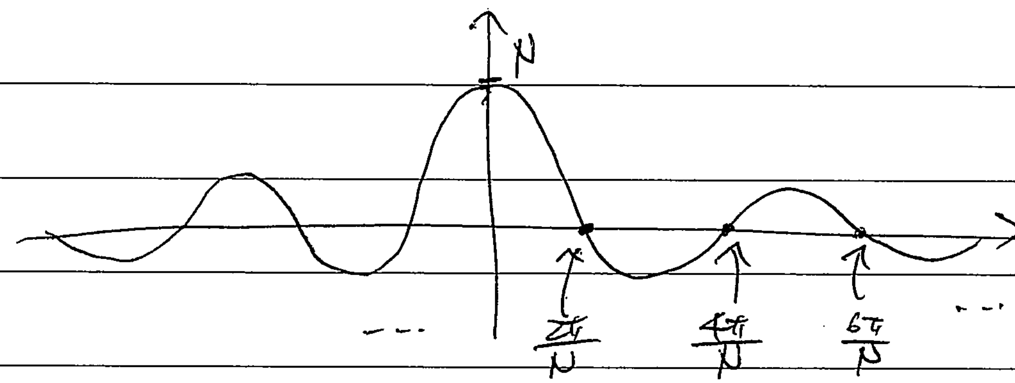
- This yields DTFT pair:

$y[n] = u[n+K] - u[n-(K+1)]$



$\overset{\text{DTFT}}{\longleftrightarrow}$ $Y(\omega) = \dfrac{\sin\left(\frac{2K+1}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$

let $N = 2K+1$



(Since $\sin(\theta) = 0$ for $\theta = m\pi$, m: integer

$\sin\left(\frac{N}{2}\omega\right) = 0$ for $\omega = m\frac{2\pi}{N}$)

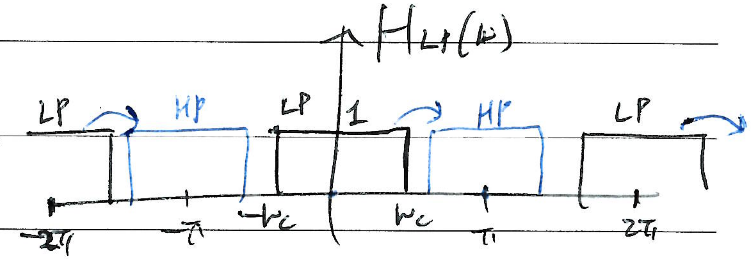
- consider

$$h_{HP}[n] = \underbrace{e^{j\pi n}}_{-1} \frac{\sin(w_c n)}{\pi n} = (-1)^n \frac{\sin(w_c n)}{\pi n}$$

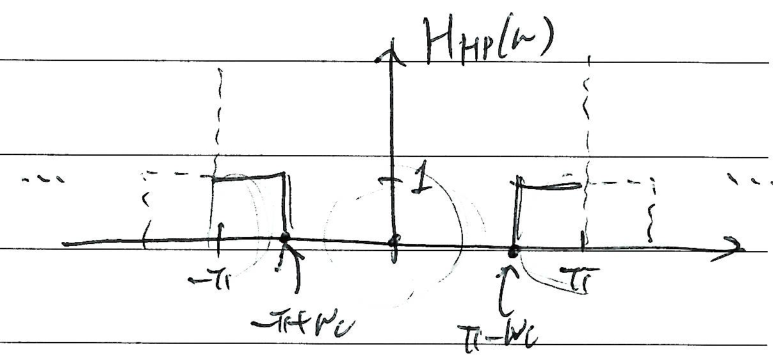
Modulation (Freq-shift) Property dictates

$$H_{HP}(w) = H_{LP}(w - \pi)$$

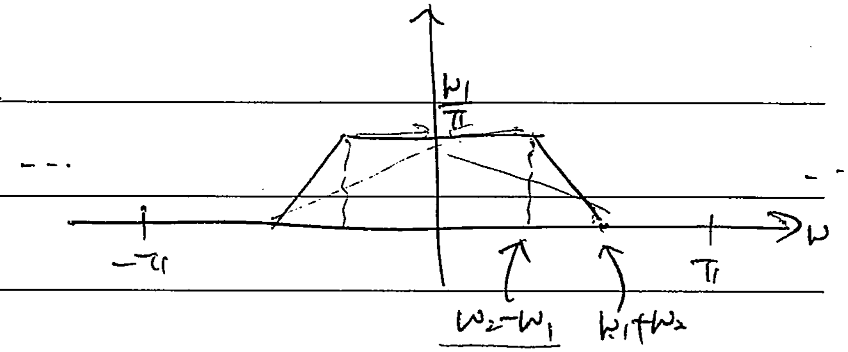
where $h_{LP}[n] = \frac{\sin(w_c n)}{\pi n} \xleftrightarrow{DTFT}$



- shifting everything over by π yields high pass filter.



$\frac{\sin(\omega_1 n)}{\pi n} \cdot \frac{\sin(\omega_2 n)}{\pi n} \overset{DTFT}{\longleftrightarrow}$

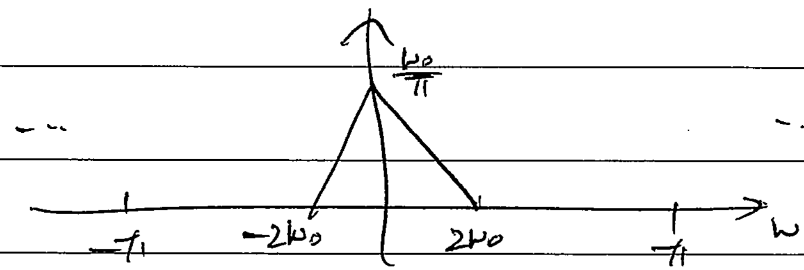


assume $\omega_1 < \omega_2$, $\omega_1+\omega_2 < \pi$.

* remember periodic with period 2π.

when $\omega_1 = \omega_2$ $(=\omega_0)$

$\left\{ \frac{\sin(\omega_0 n)}{\pi n} \right\}^2 \overset{DTFT}{\longleftrightarrow}$



assume $2\omega_0 < \pi$