Distribution Method for Two Functions of Two R.V.s

Let X, Y be r.v.s, $U = g(X,Y)$, $V = h(X,Y)$

$$F_{U,V}(u,v) = \Pr(U \le u,\ V \le v)$$
$$= \Pr(g(X,Y) \le u,\ h(X,Y) \le v)$$
$$= \iint\limits_{\substack{g(x,y) \le u \\ h(x,y) \le v}} f_{X,Y}(x,y)\, dx\, dy$$

$$f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v)$$

Note: Again $\{(x,y) : g(x,y) \le u,\ h(x,y) \le v\}$ defines a region in the xy-plane.

Jointly Gaussian R.V.s

X and Y are jointly Gaussian (JG) r.v.s if

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho_{XY}^2}}$$

$$\cdot \exp\left[\frac{-1}{2(1-\rho_{XY}^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y}\right)\right]$$

$$, -\infty < x, y < \infty \quad (\sigma_X^2, \sigma_Y^2 > 0 \ , \ |\rho_{XY}| < 1)$$

Properties of J.G. R.V.s

1) Linear Combinations of J.G. r.v.s are J.G. r.v.s

If X, Y are J.G. r.v.s $\Rightarrow$ $U = aX + bY$, $V = cX + dY$ are J.G. r.v.s.,

(a, b, c, d are constants with $ad - bc \neq 0$)

Pf: $u = ax + by$, $v = cx + dy$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Density Method

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left\| \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right\| = \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\| = ad - bc$$

Substituting in for $x(u,v)$, $y(u,v)$ and $\frac{\partial(u,v)}{\partial(x,y)}$ can get that $f_{U,V}(u,v)$ is the jointly Gaussian pdf.

2) JG r.v.s are marginally Gaussian r.v.s

X, Y JG r.v.s $\Rightarrow$ X is a Gaussian r.v.s
Y is a Gaussian r.v.s

Pf: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, \cancel{dx}\, dy$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho_{XY}^2}} \exp\left[\frac{-1}{2(1-\rho_{XY}^2)} \left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} \right)\right] dy$$

$$= \frac{1}{\sqrt{2\pi \sigma_X^2}} \exp\left(\frac{-(x-\mu_X)^2}{2\sigma_X^2}\right)$$

$$\cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma_Y^2 (1-\rho_{XY}^2)}} \exp\left[\frac{-1}{2\sigma_Y^2(1-\rho_{XY}^2)} \cdot \left(y - \mu_Y - \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x-\mu_X)\right)^2\right] dy$$

$$= \frac{1}{\sqrt{2\pi \sigma_X^2}} \exp\left(\frac{-(x-\mu_X)^2}{2\sigma_X^2}\right),$$

because integrand is a Gaussian pdf with mean $\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x-\mu_X)$ and variance $\sigma_Y^2(1-\rho_{XY}^2)$

Y can be shown to be Gaussian in a similar way.

3) JG r.v.s are conditionally Gaussian.

X, Y are JG. r.v.s. $\Rightarrow$ $f_{X|Y}(x|y)$ is Gaussian pdf

$f_{Y|X}(y|x)$ " " "

Pf: $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$

$$\Rightarrow \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\, dy$$

$$\Rightarrow \quad f_X(x) = f_X(x) \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, dy$$

But from ~~2)~~ Proof of 2) we can find $f_{Y|X}(y|x)$ from the integrand seen before.

Can write $f_{Y|X}(y|x)$ as

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma_{Y|X}^2}} \exp\left(\frac{-(y-\mu_{Y|X})^2}{2\sigma_{Y|X}^2} \right),$$

where
$$\mu_{Y|X} = \mu_Y + \frac{\sigma_Y}{\sigma_X} \rho_{XY} (x - \mu_X)$$
$$\sigma_{Y|X}^2 = \sigma_Y^2 (1 - \rho_{XY}^2)$$

Hence Y is conditionally Gaussian. Similarly show that X is also conditionally Gaussian.

4) JG r.v.s are independent if and only if they are uncorrelated.

<u>Pf</u>: If X, Y are J.G. and independent then they must uncorrelated.

If X, Y are uncorrelated, can set $\rho_{XY} = 0$ in $f_{X,Y}(x,y)$

In this case,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left(-\frac{1}{2}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right)$$

$$= f_X(x)\, f_Y(y)$$

and thus X, Y are independent.