

LAB #12

LINEARIZATION

Goal: Investigate the local behavior of a nonlinear system of differential equations near its equilibrium points by linearizing the system.

Required tools: MATLAB routine *pplane* ; eigenvalues and eigenvectors.

DISCUSSION

In the last lab (Lab # 11) you classified equilibrium points in a linear system of equations as sources, sinks, centers and saddles. (Recall, an equilibrium point is a *sink* if all solutions which begin sufficiently close to it converge to it; it is a *source* if all solutions sufficiently close to it move away from it; it is a *center* if all solutions which begin sufficiently close to it “loop around” it, i.e. they return to their initial position after a finite amount of time; the equilibrium point is a *saddle* if some solutions converge to it and some move away from it.) You also saw that for a linear system, the eigenvalues of the corresponding matrix determine what kind of equilibrium point the origin must be. In this lab, we investigate how to determine the nature of the equilibrium points of a **nonlinear** system.

ASSIGNMENT

(1) Consider first the linear system

$$\begin{cases} x' = -3x + (\sqrt{2})y \\ y' = (\sqrt{2})x - 2y \end{cases} \quad (*)$$

- Use *pplane* with window $|x| \leq 10$ and $|y| \leq 10$ to plot several orbits for the system (*). What kind of equilibrium point does the origin $(0, 0)$ seem to be? Print out your plot.
- In the *pplane* “Options” menu, select “Erase all solutions” and use *pplane* with window $|x| \leq 0.1$ and $|y| \leq 0.1$ to plot several orbits for (*). Print out this plot. Notice that all of the orbits seem to be tangent to one particular line at the origin.
- Let A be the 2×2 matrix for the system (*). As in Lab #11, use the command

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>> [B,D]=eig(A)
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to find the eigenvalues and corresponding eigenvectors for A . Use this information to prove that the behavior you observed in (a) is correct. How does the line noted in (b) relate to the eigenvectors of A ?

(2) Now consider the *nonlinear* system

$$\begin{cases} x' = -3x + (\sqrt{2})y - xy \\ y' = (\sqrt{2})x - 2y + x^2 + y^2 \end{cases} \quad (**)$$

Once again the origin $(x, y) = (0, 0)$ is an equilibrium point. Repeat parts (a) and (b) from Part (1) for this system. In part (a) plot enough orbits so that it is clear that orbits which do not begin near the origin can behave very differently than the orbits from Part (1). You should find, however, that the orbits for part (b) look very much like those for the corresponding linear system (*). Specifically, the line to which they seem to converge should be the same for both systems.

To understand why this happens, suppose that x and y are very small. We expect that the terms xy and $x^2 + y^2$ would be extremely small in comparison to x and y . In this case we expect that near $(0, 0)$, our nonlinear system (**) should behave like the linear system

$$\begin{cases} x' = -3x + (\sqrt{2})y \\ y' = (\sqrt{2})x - 2y \end{cases}$$

which is exactly the linear system (*) considered above. We refer to this system as the “**approximating linear system**” to the nonlinear system. The matrix $A = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}$ is called the “*Jacobian*” matrix at $(x, y) = (0, 0)$ for the system.

Remark: In general, for a 2 dimensional system $x' = f(x, y)$, $y' = g(x, y)$, with equilibrium point (x_1, y_1) , the associated Jacobian matrix is given by

$$J(f, g) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix},$$

evaluated at (x_1, y_1) . The approximating linear system at the equilibrium point (x_1, y_1) is then

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_1, y_1)} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}.$$

- (3) Each of the nonlinear systems (A) and (B) given below has an equilibrium point at the origin $(0, 0)$:

$$\begin{cases} x' = -2x - y - x^3 - x^2y \\ y' = x - y + x^2y + y^3 \end{cases} \quad (\mathbf{A})$$

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(0.75 - y - 0.5x) \end{cases} \quad (\mathbf{B})$$

- (a) Determine the approximating linear system for each of the nonlinear systems (A) and (B) about $(0, 0)$. Find the eigenvalues for these linear systems and use them to predict the nature of the equilibrium point at the origin for (A) and (B).

- (b) Plot some orbits for the approximating linear system to support what you said in (a) is correct. Turn in the plots.
- (c) Plot some orbits for each of the nonlinear systems to show that your predictions from (a) also apply to the nonlinear system. Turn in the plots.
- (4) A system can have an equilibrium solution which is not at the origin. For example, consider the last nonlinear system **(B)**. It is easy to check that $(0.5, 0.5)$ is another equilibrium point. Hence the constant functions $x = 0.5$ and $y = 0.5$ are also solutions to **(B)**. To study solutions near the equilibrium point $(0.5, 0.5)$, we introduce the new unknown functions $\mathbf{u} = \mathbf{x} - \mathbf{0.5}$ and $\mathbf{v} = \mathbf{y} - \mathbf{0.5}$ so that $x = u + 0.5$ and $y = v + 0.5$. This amounts to shifting the origin to the point $(0.5, 0.5)$. Substituting, we see that the above system **(B)** becomes

$$\begin{cases} u' = (u + 0.5)(1 - (u + 0.5) - (v + 0.5)) \\ v' = (v + 0.5)(0.75 - (v + 0.5) - 0.5(u + 0.5)) \end{cases}$$

or,

$$\begin{cases} u' = -0.5u - 0.5v - u^2 - uv \\ v' = -0.25u - 0.5v - v^2 - 0.5uv \end{cases} \quad (\mathbf{B}^*)$$

Hence the solutions $x = 0.5$ and $y = 0.5$ in **(B)** now correspond to $u = 0$ and $v = 0$ in **(B*)**.

- (a) Use *ppplane* to plot several orbits for solutions to **(B)** in the xy -plane. Find the approximating linear system of **(B*)** about $(0, 0)$. Plot several orbits for this linear approximation in the uv -plane. Show that when a small enough scale is used, the uv plot around the origin $(0, 0)$ looks very similar to the xy plot around $(0.5, 0.5)$. Turn in your plots.
- (b) Use MATLAB to find the eigenvalues and corresponding eigenvectors of the approximating linear system of **(B*)** about $(0, 0)$ you found in (a). Then, using the general solution of the system, prove that the origin is a sink.
- (5) Our examples so far seem to indicate that the solutions to a nonlinear system always behave very much like those of the corresponding linear system as long as we stay near the equilibrium point. This is not always true. For example, consider the nonlinear system below:

$$\begin{cases} x' = -y - (\text{seed})x^3 \\ y' = x \end{cases} \quad (\mathbf{C})$$

- (a) Let x and y be solutions to the linear approximation of **(C)**, then they satisfy

$$\begin{cases} x' = -y \\ y' = x \end{cases} \quad (\mathbf{D})$$

Plot a few orbits of the corresponding linear approximation. They should appear to be circles around the origin. Prove that they really are circular. This can be done without solving the system as follows.

Note that

$$(x^2 + y^2)' = 2xx' + 2yy' \quad (\mathbf{E})$$

and then use the system **(D)** to show that the expression **(E)** is 0. How does it follow that the orbits of the system **(D)** are circles ?

- (b) Use ***pplane*** to plot some orbits of the system **(C)** over the intervals $|x| \leq 2$ and $|y| \leq 2$. The orbits should seem to spiral in toward the origin, but with a blank spot near the origin.
- (c) Change to a smaller scale in ***pplane*** ($|x| \leq 0.0002$, $|y| \leq 0.0002$) and plot some orbits so that you can see what is happening around the origin for the system **(C)**. Near the origin the orbits should appear to be loops. Actually, this is not at all what is happening. Use **(C)** to prove that if $x(t)$ and $y(t)$ are solutions to the system **(C)**, then

$$(x^2 + y^2)' = 2xx' + 2yy' = -2(\text{seed})x^4 \quad (***)$$

How does it follow that the orbits constantly move toward the origin ? (*Hint*: How does the sign of the derivative of a function relate to the behavior of the function ?) How can you explain the fact that the computer output seems to suggest that the orbits are loops while the theory proves that they are spirals?

- (d) From the equation (***) in (c), you proved that *every* non-equilibrium orbit moves constantly towards the origin. However, their graphs in ***pplane*** seem to indicate that they do not approach the origin but instead each orbit seems to approach a circle of finite radius: Use ***pplane*** to plot the specific orbit which starts at $x = 0$, $y = -0.2$, use the intervals $|x| \leq 0.3$ and $|y| \leq 0.3$ and use the “*Zoom In*” feature several times near the origin to approximate this radius. But of course no orbits approach any circle of positive radius !

Remark: The examples in this lab demonstrate a theorem concerning the local behavior of a nonlinear system of equations near its equilibrium points to the corresponding linearized system. If $f(x, y)$ and $g(x, y)$ are polynomials in x and y , with $f(0, 0) = g(0, 0) = 0$, then the system

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

will have the same type of equilibrium point at $(0, 0)$ (source, sink or saddle) as the corresponding linear system as long as the eigenvalues of the linear system are distinct and are not purely imaginary i.e., $(0, 0)$ is not an improper node (repeated eigenvalue with one independent eigenvector) nor a center of the linearized system.