Invertibility and DT LTI systems

$$x[n] \rightarrow \boxed{\text{system}} \xrightarrow{y[n]} \boxed{\text{Inverse system}} \rightarrow w[n] = x[n]$$, for any $x[n]$

- If the system is LTI, we have

$$x[n] \rightarrow \boxed{h[n]} \xrightarrow{y[n]} \boxed{h_I[n]} \rightarrow w[n] = x[n]$$

- Two LTI systems in series can be replaced by a single LTI system with impulse respone

$$x[n] \rightarrow \boxed{h_o[n] = h[n] * h_I[n]} \rightarrow w[n] = x[n]$$

↳ $x[n] * \underline{(h[n] * h_I[n])} = x[n]$

- At the same time, we know $\underline{x[n]} * \underline{\delta[n]} = \underline{x[n]}$

Thus, if the inverse system exists for an LTI system, it must satisfy $\underline{\underline{h[n] * h_I[n] = \delta[n]}}$

"AI = A"

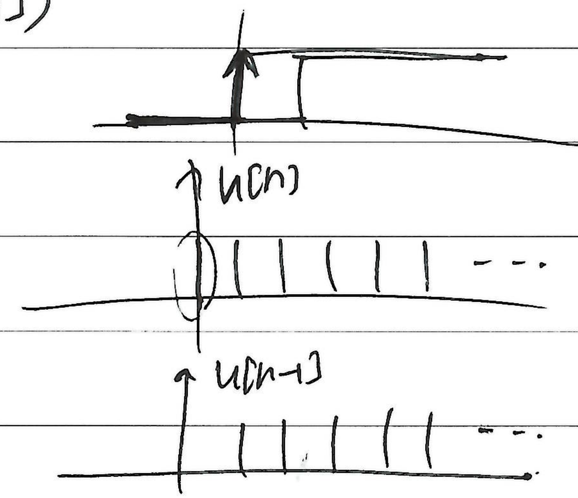
e.g. $h[n] = u[n]$ and $h_I[n] = \delta[n] - \delta[n-1]$

$$\underline{h[n] * h_I[n]} = u[n] * (\delta[n] - \delta[n-1])$$

$$= u[n] - u[n-1]$$

$$= \underline{\delta[n]}$$

Inverse checked!



- Similarly for a CT LTI system with impulse response $h(t)$, if the inverse system exists, it must satisfy:

$$h(t) * h_I(t) = \delta(t)$$

where $h_I(t)$ is the impulse response of the inverse system.

DT	CT
"$\underline{x[n]}$",	"$\underline{x(t)}$"

- BIBO Stability (necessary cond.)

If $\sum_{n=-\infty}^{\infty} |h[n]| < \infty \underset{\not\leftarrow}{\overset{?}{\rightarrow}}$ then system is BIBO stable.

(proof) $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

Bounded input: $|x[n]| < A_{max} < \infty$, $\forall n$

Using Triangle Inequality: $\begin{cases} |a+b| \le |a| + |b| \\ |ab| \le |a||b| \end{cases}$

$|y[n]| < \sum_k |h[k]x[n-k]| = \sum |h[k]|\,|x[n-k]|$

Since $|x[n-k]| < A_{max}$, $\forall n, \forall k$

It follows, $|y[n]| < \{\sum_k |h[k]|\} \times A_{max}$

Thus, if $B = \sum_k |h[k]| < \infty$, then $|y[n]| < \infty$

- Causality: $y[n] = \sum_k h[k]\, x[n-k]$

$$\cancel{= \cdots} + \cancel{h[-2]\, \underline{x[n+2]}} + \cancel{h[-1]\, \underline{x[n+1]}} + h[0]\, \underline{x[n]} + h[1]\, \underline{x[n-1]} + \cdots$$

Thus, if $\underline{h[n] = 0 \text{ for } n < 0}$, then system is causal (does not depend on future ~~values~~ inputs.)

Chap 4. Continuous-Time Fourier Transform

- Consider sinewave input to LTI system

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(\omega_0)\, e^{j\omega_0 t}$$

$$\left(\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega_0 (t-\tau)}\, d\tau \\ &= \underbrace{\left\{ \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0 \tau}\, d\tau \right\}}_{= H(\omega_0)} e^{j\omega_0 t} = H(\omega_0)\, e^{j\omega_0 t} \end{aligned} \right)$$

where ~~$H(\omega_0)$~~ $H(\omega)$ is the Fourier Transform of the impulse response defined as

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

Def CT FT (Fourier Transform)

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

- $X(\omega)$ reflects how the energy of a signal is distributed as a func. of freq.

We will prove Parseval's Theorem later

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

→ other useful info. about the signal is gleaned from $X(\omega)$ which is generally complex-valued for energy freq ω.

<Notation & Inverse FT>

"$X(\omega) = \mathcal{F}\{x(t)\}$"

* This mapping is one-to-one ⇒ uniqueness"

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$x(t)$ time-domain

$$\boxed{x(t) \xleftrightarrow{\mathcal{F}} X(\omega)}$$

$X(\omega)$ freq.-domain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

"$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$"

Notation Issue.

(Recall) Laplace Transform : $X(s) = \int_0^\infty x(t) e^{-st} dt$

(one-sided Laplace Transform) ← good for dealing with initial conditions.

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

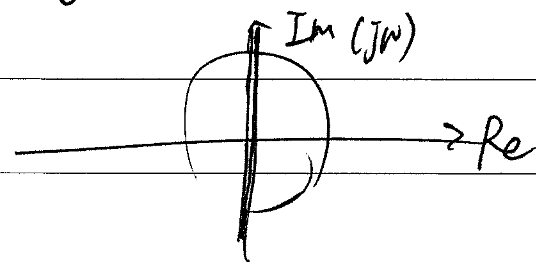
(Two-sided Laplace Transform)

Many textbooks make connection btw Laplace and Fourier Transform as

$$X(j\omega) = X(s)\Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

That is, Fourier Transform is the Laplace transform evaluated along the imaginary axis in the s-plane.

$s = \sigma + j\omega$



→ We will not use this notation. ($X(j\omega)$) $\rightarrow$ $X(\omega)$

since it's awkward and confusing.

We will define Fourier Transform as if Laplace Transform is/was not even defined.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = |X(\omega)|\, e^{j\angle X(\omega)}$$

is generally complex-valued at every freq. ω.

- The phase of $X(\omega)$, $\angle X(\omega)$, allows us to distinguish between the FT of $x(t)$ and that of $x(t-t_0)$
  (energy distribution as a function of ω is still the same)
- The phase of $X(\omega)$ also allows us to distinguish between the FT of $\cos(\omega_0 t)$ and that of $\sin(\omega_0 t)$
  (" " ")
  energy is concentrated at ω_0 and $-\omega_0$: both.

- Remember the uniqueness of the FT 1-to-1 mapping btw time domain and freq domain.

(e.g. 4.1 on page 290)

$$x(t) = e^{-at}u(t)$$

$$X(\omega) = \int_0^\infty e^{-at} e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^\infty = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega}$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega} \leftarrow X(\omega)$$

Re (a)
Im (ω)

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

(see plots in Fig 4.5)