Real Analysis Qual Prep   Summer 2009

Assignment 5: L^p Spaces

1. Let $(X, \mathcal{F}, \mu)$ be a measure space, $f \in L^p(\mu)$, $1 \leq p \leq \infty$. Suppose there exist sets E_n satisfying $\mu(E_n) = 1/n$ for all n. Show

$$\lim_{n\to\infty} (n^{\frac{p-1}{p}} \int_{E_n} |f| d\mu) = 0$$

2. Verify that for every measurable function f on a sigma-finite measure space $(X, \mathcal{F}, \mu)$, and $1 \leq p < \infty$,

$$\int_X |f|^p d\mu = \int_0^\infty pt^{p-1} \mu \{|f| > t\}\, dt$$

3. If $f \geq 0$, show that

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x) dt$$

4. Let $I = [0, \pi]$. Show that $\int_I x^{-1/4} \sin(x) dx \leq \pi^{3/4}$.

5. Let $I = [0, \pi]$ and $f \in L^2(I)$. Is it possible to have simultaneously

$$\int_I (f(x) - \sin(x))^2 dx \leq 4/9$$

and

$$\int_I (f(x) - \cos(x))^2 dx \leq 1/9?$$

6. Find an example of a proper non-trivial closed subspace of $L^2([0, 1])$ and an example of a subspace of $L^2([0, 1])$ that is not closed.

7. Let $(X, \mathcal{F}, \mu)$ be a measure space. Find all functions $f : X \to [0, \infty)$ satisfying

$$||f||_p^p = ||f||_1 < \infty$$

for all $p > 0$.

8. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n : X \to [0, \infty)$ be such that $||f_n||_p \leq 1$, $1 < p < \infty$, and $f_n \to f$ a.e. Show that $f \in L^p(\mu)$ and $||f_n - f||_1 \to 0$.

9. True or false: If $f_n \in L^1([0, 1])$ and $f_n \to 0$ in L^1, then $f_n \to 0$ a.e.

10. Let $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$. Compute

$$\lim_{h\to 0} \int_{\mathbf{R}^n} |f(x + h) - f(x)|^p dx$$

11. Assume $1 < p < \infty$, $1/p + 1/q = 1$, $f \in L^p$, $g \in L^q$.

   (a) For $x \in \mathbf{R}$, let $K_x(y) = f(x-y)g(y)$. Show that $K_x \in L^1$.

   (b) Let $h(x) = \int f(x-y)g(y)dy$. Show that h is bounded.

   (c) Show h is continuous.

12. Let $(X, \mathcal{F}, \mu)$ be a sigma-finite measure space, $1 \leq p_1, p_2 < \infty$. Suppose there exist constants c_1, c_2 such that

$$\mu\{x : |f(x)| > y\} \leq \tfrac{c_j}{p_j},\ j = 1, 2, \text{ for all } y > 0.$$

   Show that $f \in L^p(\mu)$, $p_1 < p < p_2$. Hint: Use Problem 2.

13. Let $(X, \mathcal{F}, \mu)$ be a finite measure space, $1 < p < \infty$. Suppose $f_n \to f$ a.e., $||f_n||_p \leq 1$ for all n. Show

$$\textstyle\int_X f_n g d\mu \to \int_X f g d\mu$$

   , for all $g \in L^q(\mu)$, $1/p + 1/q = 1$.

14. Let $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and let $f_0(x) = xf(x)$. Show that

$$||f||_1 \leq (8||f||_2||f_0||_2)^{1/2}.$$

15. Let $(X, \mathcal{F}, \mu)$ be a measure space, $1 < p < \infty$. If f_n, $f \in L^p(\mu)$ and $\int_X f_n g d\mu \to \int_X f g d\mu$ for every $g \in L^q(\mu)$, $1/p + 1/q = 1$, show that

$$||f||_p \leq \liminf ||f_n||_p.$$