1.

(a) $x[n]$ periodic with period $N = 12$

$$X_k = \frac{1}{12}\sum_{n=0}^{11} x[n]\, e^{-j2\pi kn/12} = \frac{1}{12}\sum_{n=-1}^{10} x[n]\, e^{-j\pi kn/6}$$

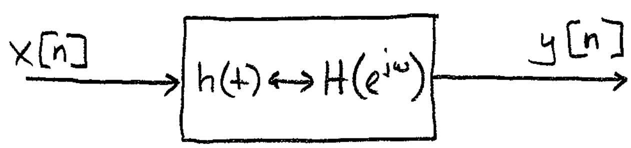
$$= \frac{1}{12} e^{+j\pi k/6} + \frac{1}{12} + \frac{1}{12} e^{-j\pi k/6}$$

$$= \frac{1}{12} + \frac{1}{6}\cos(\pi k/6).$$

In general if $x[n] \longleftrightarrow X_k$ (DTFS pair) then the DTFT of $x[n]$ is

$$X(e^{j\omega}) = 2\pi \sum_{\ell=-\infty}^{\infty} X_\ell\, \delta\!\left(\omega - \frac{2\pi \ell}{N}\right) \qquad (N = \text{period})$$

Thus if



then

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$= 2\pi\, H(e^{j\omega}) \sum_{\ell=-\infty}^{\infty} X_\ell\, \delta\!\left(\omega - \frac{2\pi\ell}{N}\right)$$

$$= 2\pi \sum_{\ell=-\infty}^{\infty} H(e^{j2\pi\ell/N})\, X_\ell\, \delta\!\left(\omega - \frac{2\pi\ell}{N}\right).$$

Thus if we define

$$Y_\ell = H(e^{j2\pi\ell/N})\, X_\ell$$

we see that Y_ℓ must be periodic in ℓ with period N (because X_ℓ is N-periodic and $H(e^{j\omega})$ is periodic in ω of period 2π).

$\Longrightarrow$ $y[n]$ must be N-periodic (b-i) Since $y_1[n]$ is periodic with period 6, it is also periodic with period 12

$\Longrightarrow$ It could be the output of LTI system having input $x[n]$.

(b-ii)
The LTI system would have to satisfy

$$H(e^{j\pi \ell/6}) = \frac{Y_\ell}{X_\ell} \qquad \ell = 0, 1, 2, \ldots 11.$$

provided that $X_\ell \neq 0$. If $X_\ell = 0$ then must have $Y_\ell = 0$ and

$$H(e^{j\pi \ell/6}) = \text{"don't care"}$$

Thus must check this to be safe (see following pages)

I did not really expect you to check this on the test.

In general, the $H(e^{j\omega})$ required is <u>not</u> unique since we are only constraining certain of its samples. (Can you think of an additional condition on $h[n]$ which would make the LTI system unique?)

(c-i) Now $y_2[n]$ is periodic with period 9. It is <u>not</u> also periodic with period 12.

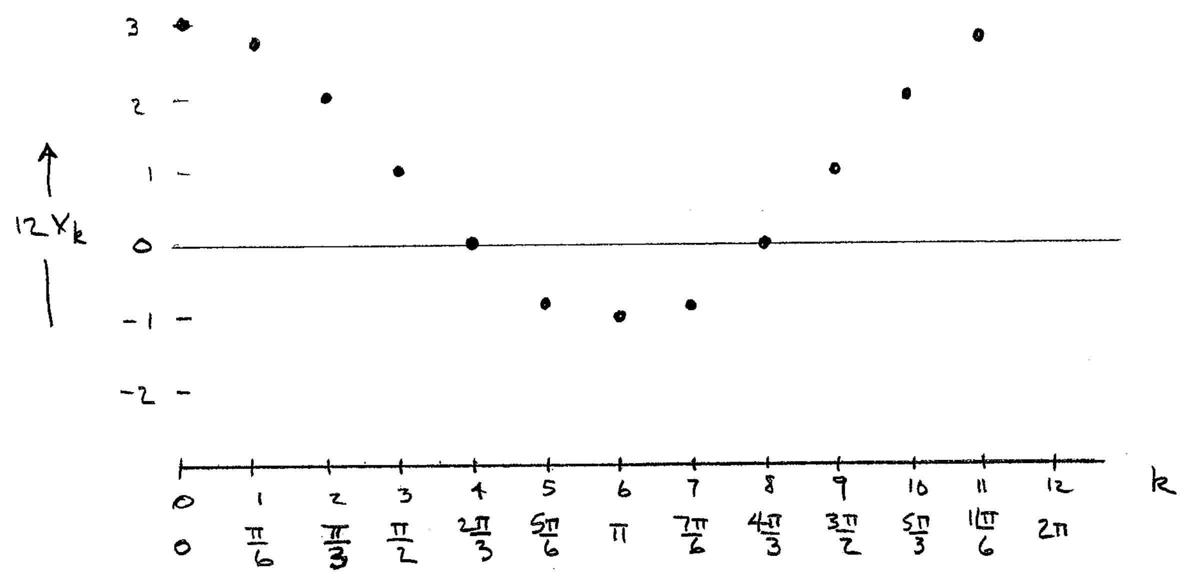
Hence <u>there is no</u> LTI system with input $x[n]$ and output $y_2[n]$.

(c-ii) N.A.

$\boxed{\text{CHECKING THAT } X_k = 0 \Rightarrow Y_k = 0}$

Must be certain that $Y_k = 0$ if $X_k = 0$. Thus need to find indices of these zero frequencies. From (a)

$$X_k = \frac{1}{12}\left(1 + 2\cos\frac{\pi k}{6}\right) \qquad k = 0, 1, \ldots, 11.$$



$$\Longrightarrow X_4 = X_8 = 0$$

If $y_1[n]$ comes from LTI driven by $x[n]$ we must have $Y_{1,4} = Y_{1,8} = 0$ where

$$Y_{1,k} = \frac{1}{12}\sum_{n=0}^{11} y_1[n]\, e^{-j\pi k n/6}$$

$$= \frac{1}{12}\sum_{n=-4}^{-2} e^{-j\pi k n/6} + \frac{1}{12}\sum_{n=2}^{4} e^{-j\pi k n/6}$$

$$= \frac{1}{12}\left\{ e^{+j4\pi k/6} + e^{j3\pi k/6} + e^{j2\pi k/6} + e^{-j2\pi k/6} + e^{-j3\pi k/6} + e^{-j4\pi k/6} \right\}$$

$$= \frac{1}{6}\left\{ \cos\left(\frac{2\pi k}{3}\right) + \cos\left(\frac{\pi k}{2}\right) + \cos\left(\frac{\pi k}{3}\right) \right\}.$$

$$\therefore \quad Y_{1,4} = \frac{1}{6}\left\{ \cos\frac{8\pi}{3} + \cos 2\pi + \cos\frac{4\pi}{3} \right\} = 0$$

$$Y_{1,8} = \frac{1}{6}\left\{ \cos\frac{16\pi}{3} + \cos 4\pi + \cos\frac{8\pi}{3} \right\} = 0$$

$\therefore$ No problem.

2.

(a) $f[n] = e^{-3n}\cos(3n)u[n]$. Find $F(e^{j\omega})$.

Can use transform tables and properties to compute (also not hard to directly compute).

$$\tilde{f}[n] \triangleq e^{-3n}u[n] = \left(e^{-3}\right)^n u[n]$$

$$\updownarrow$$

$$\tilde{F}(e^{j\omega}) = \frac{1}{1-e^{-3}e^{-j\omega}}$$

Also $\cos(3n) = \frac{e^{j3n}+e^{-j3n}}{2}$ and use the modulation or frequency shifting property to conclude

$$f[n] = \tfrac{1}{2}e^{j3n}\tilde{f}[n] + \tfrac{1}{2}e^{-j3n}\tilde{f}[n]$$

$$\updownarrow$$

$$F(e^{j\omega}) = \tfrac{1}{2}\tilde{F}\left(e^{j(\omega-3)}\right) + \tfrac{1}{2}\tilde{F}\left(e^{j(\omega+3)}\right)$$

$$= \frac{1/2}{1-e^{-3}e^{-j(\omega-3)}} + \frac{1/2}{1-e^{-3}e^{-j(\omega+3)}}$$

$$= \frac{\tfrac{1}{2}\left(1-e^{-3}e^{-j(\omega+3)}\right)+\tfrac{1}{2}\left(1-e^{-3}e^{-j(\omega-3)}\right)}{1-e^{-3}e^{-j(\omega+3)}-e^{-3}e^{-j(\omega-3)}+e^{-6}e^{-j(\omega-3)}e^{-j(\omega+3)}}$$

$$= \frac{1-\tfrac{1}{2}e^{-3}e^{-j3}e^{-j\omega}-\tfrac{1}{2}e^{-3}e^{+j3}e^{-j\omega}}{1-e^{-3}e^{-j3}e^{-j\omega}-e^{-3}e^{+j3}e^{-j\omega}+e^{-6}e^{-j2\omega}}$$

$$= \frac{1-\tfrac{1}{2}e^{-3}e^{-j\omega}\left(e^{-j3}+e^{+j3}\right)}{1-e^{-3}e^{-j\omega}\left(e^{-j3}+e^{+j3}\right)+e^{-6}e^{-j2\omega}}$$

$$= \frac{1-e^{-3}\cos(3)e^{-j\omega}}{1-2e^{-3}\cos(3)e^{-j\omega}+e^{-6}e^{-j2\omega}}$$

(b) $X(e^{j\omega}) = \dfrac{e^{-j2\omega}-1}{e^{-j2\omega}-4} \longleftrightarrow x[n]$. Find $x[n]$.

Let $v = e^{-j\omega}$. Then

$$X(v) = \frac{v^2-1}{v^2-4}$$

$$\begin{array}{r|l} & 1 \\ \hline v^2-4 & v^2-1 \\ & v^2-4 \\ \hline & \quad 3 \end{array}$$

$$= 1 + \frac{3}{v^2-4}$$

$$= 1 + \frac{3}{(v-2)(v+2)} = 1 - \frac{3/4}{(1-\frac{1}{2}v)(1+\frac{1}{2}v)}.$$

Do a partial fraction expansion

$$\frac{3/4}{(1-\frac{1}{2}v)(1+\frac{1}{2}v)} = \frac{A}{1-\frac{1}{2}v} + \frac{B}{1+\frac{1}{2}v}.$$

$$A = \left.\frac{3/4}{1+\frac{1}{2}v}\right|_{v=2} = \frac{3/4}{2} = \frac{3}{8}$$

$$B = \left.\frac{3/4}{1-\frac{1}{2}v}\right|_{v=-2} = \frac{3/4}{2} = \frac{3}{8}$$

$$\therefore\ X(e^{j\omega}) = 1 - \frac{3/8}{1-\frac{1}{2}e^{-j\omega}} - \frac{3/8}{1+\frac{1}{2}e^{-j\omega}}$$

$$\updownarrow$$

$$x[n] = \delta[n] - \frac{3}{8}\left(\frac{1}{2}\right)^n u[n] - \frac{3}{8}\left(-\frac{1}{2}\right)^n u[n]$$

(c) Find energy in $y[n] \longleftrightarrow Y(e^{j\omega}) = \dfrac{1-ae^{-j\omega}}{a-e^{-j\omega}}$

$$\left(\text{Energy in } y[\cdot]\right) = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(e^{j\omega})|^2\, d\omega.$$

$$|Y(e^{j\omega})|^2 = \frac{1-ae^{-j\omega}}{a-e^{-j\omega}}\ \frac{1-a\,e^{j\omega}}{a-e^{j\omega}}$$

$$= \frac{1-ae^{-j\omega}-a\,e^{j\omega}+a^2}{a^2-ae^{-j\omega}-ae^{j\omega}+1}$$

$$|Y(e^{j\omega})|^2 = \frac{1+a^2-2a\cos(\omega)}{1+a^2-2a\cos(\omega)} = 1 \quad \forall \omega.$$

$$\therefore \text{ energy } = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \, d\omega = 1.$$

3.

(a) $r(t) = \sum_k (-1)^k \delta(t - kT/2)$

$$= \sum_n \delta(t-nT) - \sum_m \delta(t-\tfrac{T}{2}-mT)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\frac{2\pi}{T}\sum_k \delta(\omega - \tfrac{2\pi k}{T}) - \frac{2\pi}{T} e^{-j\omega T/2} \sum_k \delta(\omega - \tfrac{2\pi k}{T}).$$

$$\therefore R(j\omega) = \frac{2\pi}{T}\sum_k \left\{ \delta(\omega - \tfrac{2\pi k}{T}) - \underbrace{e^{-j\frac{2\pi k}{T}\frac{T}{2}}}_{e^{-j\pi k} = (-1)^k} \delta(\omega - \tfrac{2\pi k}{T}) \right\}$$

$$= \frac{2\pi}{T}\sum_k \underbrace{(1-(-1)^k)}_{= \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}} \delta(\omega - \tfrac{2\pi k}{T})$$

$$= \frac{4\pi}{T}\sum_\ell \delta(\omega - \tfrac{2\pi}{T} - \tfrac{4\pi\ell}{T})$$ Sketch next page.

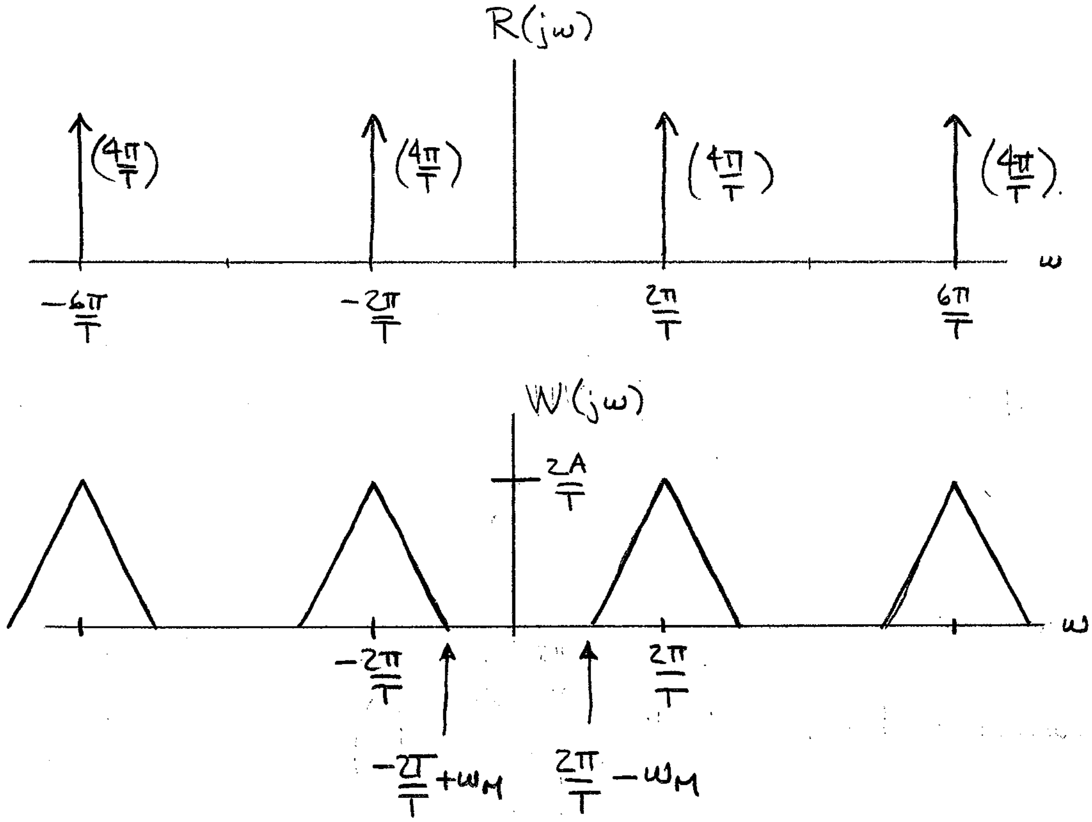
(b) $w(t) = r(t)x(t) \longleftrightarrow W(j\omega) = \frac{1}{2\pi} X(j\omega) * R(j\omega)$

$$\therefore W(j\omega) = \frac{2}{T}\sum_\ell X\left(j\left(\omega - \tfrac{2\pi}{T} - \tfrac{4\pi\ell}{T}\right)\right)$$

(c) From plot condition is

$$\frac{2\pi}{T} - \omega_M > 0$$

$$\frac{2\pi}{T} > \omega_M$$


R(jω)
(4π/T)
(4π/T)
(4π/T)
(4π/T)
ω
−6π/T
−2π/T
2π/T
6π/T
W(jω)
2A/T
ω
−2π/T
2π/T
−2π/T + ω_M
2π/T − ω_M

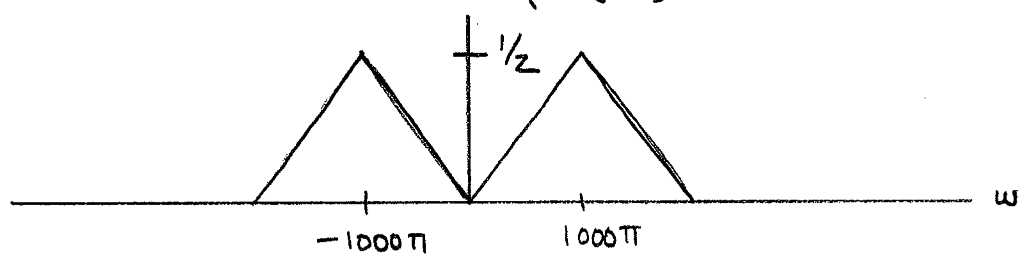
4.

(a) $g(t) = x(t)\cos(1000\pi t)$

$\updownarrow$

$$G(j\omega) = \frac{1}{2\pi} X(j\omega) * \pi\left[\delta(\omega - 1000\pi) + \delta(\omega + 1000\pi)\right]$$

$$= \frac{1}{2} X(j(\omega - 1000\pi)) + \frac{1}{2} X(j(\omega + 1000\pi))$$



$(\mathrm{Im}\{G(j\omega)\} = 0)$.

(b) $w(t) = g(t)\sin(1000\pi t)$

$$= x(t)\cos(1000\pi t)\sin(1000\pi t)$$

$$= \frac{1}{2} x(t)\sin(0) + \frac{1}{2} x(t)\sin(2000\pi t)$$

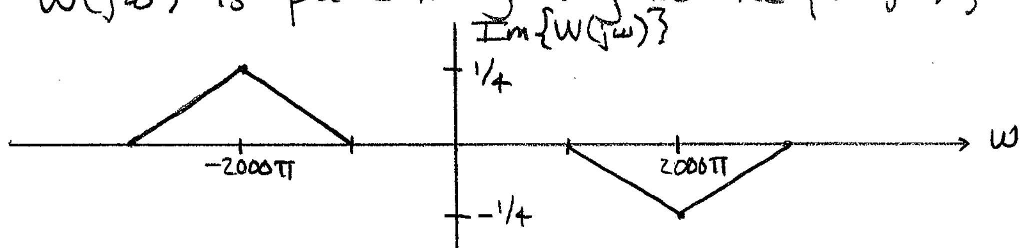
$$= \frac{1}{2} x(t)\sin(2000\pi t).$$

$$\therefore\ W(j\omega) = \frac{1}{2}\,\frac{1}{2\pi} X(j\omega) * \frac{\pi}{j}\left[\delta(\omega - 2000\pi) - \delta(\omega + 2000\pi)\right]$$

$$= \frac{1}{4j} X(j(\omega - 2000\pi)) - \frac{1}{4j} X(j(\omega + 2000\pi))$$

$$= j\left\{\frac{1}{4} X(j(\omega + 2000\pi)) - \frac{1}{4} X(j(\omega - 2000\pi))\right.$$

$\Longrightarrow$ $W(j\omega)$ is pure imaginary ie $\mathrm{Re}\{W(j\omega)\} = 0$



(c) $y(t) = 0$.