of harmonically related complex exponentials that share a common period with the signal being represented. In addition, we have seen that the Fourier series representation has a number of important properties which describe how different characteristics of signals are reflected in their Fourier series coefficients.

One of the most important properties of Fourier series is a direct consequence of the eigenfunction property of complex exponentials. Specifically, if a periodic signal is applied to an LTI system, then the output will be periodic with the same period, and each of the Fourier coefficients of the output is the corresponding Fourier coefficient of the input multiplied by a complex number whose value is a function of the frequency corresponding to that Fourier coefficient. This function of frequency is characteristic of the LTI system and is referred to as the frequency response of the system. By examining the frequency response, we were led directly to the idea of filtering of signals using LTI systems, a concept that has numerous applications, including several that we have described. One important class of applications involves the notion of frequency-selective filtering—i.e., the idea of using an LTI system to pass certain specified bands of frequencies and stop or significantly attentuate others. We introduced the concept of ideal frequency-selective filters and also gave several examples of frequency-selective filters described by linear constant-coefficient differential or difference equations.

The purpose of this chapter has been to begin the process of developing both the tools of Fourier analysis and an appreciation for the utility of these tools in applications. In the chapters that follow, we continue with this agenda by developing the Fourier transform representations for aperiodic signals in continuous and discrete time and by taking a deeper look not only at filtering, but also at other important applications of Fourier methods.

Chapter 3 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

3.1. A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T = 8$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2, a_3 = a_{-3}^* = 4j.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k).$$

3.2. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N = 5$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 1, a_2 = a_{-2}^* = e^{j\pi/4}, a_4 = a_{-4}^* = 2e^{j\pi/3}.$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(\omega_k n + \phi_k).$$

3.3. For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

3.4. Use the Fourier series analysis equation (3.39) to calculate the coefficients a_k for the continuous-time periodic signal

$$x(t) = \begin{cases} 1.5, & 0 \le t < 1 \\ -1.5, & 1 \le t < 2 \end{cases}$$

with fundamental frequency $\omega_0 = \pi$.

3.5. Let $x_1(t)$ be a continuous-time periodic signal with fundamental frequency ω_1 and Fourier coefficients a_k. Given that

$$x_2(t) = x_1(1-t) + x_1(t-1),$$

how is the fundamental frequency ω_2 of $x_2(t)$ related to ω_1? Also, find a relationship between the Fourier series coefficients b_k of $x_2(t)$ and the coefficients a_k. You may use the properties listed in Table 3.1.

3.6. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t},$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk\frac{2\pi}{50}t},$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t}.$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are real valued?

(b) Which of the three signals is/are even?

3.7. Suppose the periodic signal $x(t)$ has fundamental period T and Fourier coefficients a_k. In a variety of situations, it is easier to calculate the Fourier series coefficients

b_k for $g(t) = dx(t)/dt$, as opposed to calculating a_k directly. Given that

$$\int_T^{2T} x(t)\,dt = 2,$$

find an expression for a_k in terms of b_k and T. You may use any of the properties listed in Table 3.1 to help find the expression.

3.8. Suppose we are given the following information about a signal $x(t)$:

1. $x(t)$ is real and odd.
2. $x(t)$ is periodic with period $T = 2$ and has Fourier coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. $\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1$.

Specify two different signals that satisfy these conditions.

3.9. Use the analysis equation (3.95) to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

$$x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n-4m] + 8\delta[n-1-4m]\}.$$

3.10. Let $x[n]$ be a real and odd periodic signal with period $N = 7$ and Fourier coefficients a_k. Given that

$$a_{15} = j,\ a_{16} = 2j,\ a_{17} = 3j,$$

determine the values of a_0, a_{-1}, a_{-2}, and a_{-3}.

3.11. Suppose we are given the following information about a signal $x[n]$:

1. $x[n]$ is a real and even signal.
2. $x[n]$ has period $N = 10$ and Fourier coefficients a_k.
3. $a_{11} = 5$.
4. $\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = 50$.

Show that $x[n] = A\cos(Bn + C)$, and specify numerical values for the constants A, B, and C.

3.12. Each of the two sequences $x_1[n]$ and $x_2[n]$ has a period $N = 4$, and the corresponding Fourier series coefficients are specified as

$$x_1[n] \longleftrightarrow a_k, \quad x_2[n] \longleftrightarrow b_k,$$

where

$$a_0 = a_3 = \frac{1}{2}a_1 = \frac{1}{2}a_2 = 1 \quad \text{and} \quad b_0 = b_1 = b_2 = b_3 = 1.$$

Using the multiplication property in Table 3.1, determine the Fourier series coefficients c_k for the signal $g[n] = x_1[n]x_2[n]$.

3.13. Consider a continuous-time LTI system whose frequency response is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \frac{\sin(4\omega)}{\omega}.$$

If the input to this system is a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

with period $T = 8$, determine the corresponding system output $y(t)$.

3.14. When the impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$$

is the input to a particular LTI system with frequency response $H(e^{j\omega})$, the output of the system is found to be

$$y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right).$$

Determine the values of $H(e^{jk\pi/2})$ for $k = 0, 1, 2,$ and 3.

3.15. Consider a continuous-time ideal lowpass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \le 100 \\ 0, & |\omega| > 100 \end{cases}.$$

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$$x(t) \xrightarrow{S} y(t) = x(t).$$

For what values of k is it guaranteed that $a_k = 0$?

3.16. Determine the output of the filter shown in Figure P3.16 for the following periodic inputs:

(a) $x_1[n] = (-1)^n$

(b) $x_2[n] = 1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right)$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}\left(\frac{1}{2}\right)^{n-4k} u[n - 4k]$

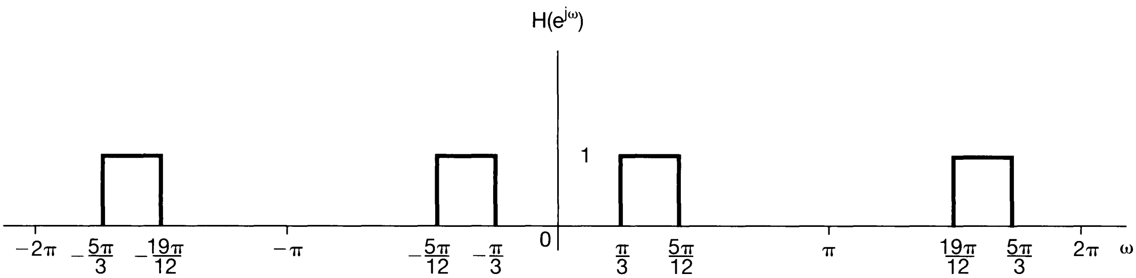


Figure P3.16

3.17. Consider three continuous-time systems S_1, S_2, and S_3 whose responses to a complex exponential input e^{j5t} are specified as

$$S_1 : e^{j5t} \longrightarrow te^{j5t},$$
$$S_2 : e^{j5t} \longrightarrow e^{j5(t-1)},$$
$$S_3 : e^{j5t} \longrightarrow \cos(5t).$$

For each system, determine whether the given information is sufficient to conclude that the system is definitely *not* LTI.

3.18. Consider three discrete-time systems S_1, S_2, and S_3 whose respective responses to a complex exponential input $e^{j\pi n/2}$ are specified as

$$S_1 : e^{j\pi n/2} \longrightarrow e^{j\pi n/2}u[n],$$
$$S_2 : e^{j\pi n/2} \longrightarrow e^{j3\pi n/2},$$
$$S_3 : e^{j\pi n/2} \longrightarrow 2e^{j5\pi n/2}.$$

For each system, determine whether the given information is sufficient to conclude that the system is definitely *not* LTI.

3.19. Consider a causal LTI system implemented as the *RL* circuit shown in Figure P3.19. A current source produces an input current $x(t)$, and the system output is considered to be the current $y(t)$ flowing through the inductor.

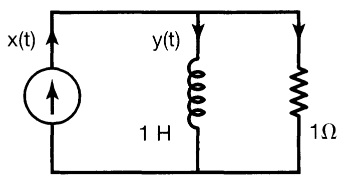


Figure P3.19

(a) Find the differential equation relating $x(t)$ and $y(t)$.
(b) Determine the frequency response of this system by considering the output of the system to inputs of the form $x(t) = e^{j\omega t}$.
(c) Determine the output $y(t)$ if $x(t) = \cos(t)$.

3.20. Consider a causal LTI system implemented as the *RLC* circuit shown in Figure P3.20. In this circuit, $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered the system output.

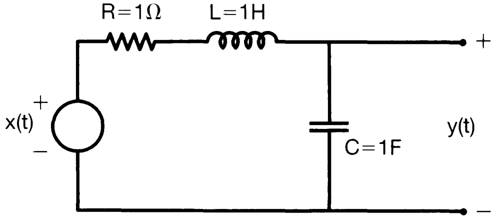


Figure P3.20

(a) Find the differential equation relating $x(t)$ and $y(t)$.
(b) Determine the frequency response of this system by considering the output of the system to inputs of the form $x(t) = e^{j\omega t}$.
(c) Determine the output $y(t)$ if $x(t) = \sin(t)$.

BASIC PROBLEMS

3.21. A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T = 8$. The nonzero Fourier series coefficients for $x(t)$ are specified as

$$a_1 = a_{-1}^* = j, a_5 = a_{-5} = 2.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

3.22. Determine the Fourier series representations for the following signals:
(a) Each $x(t)$ illustrated in Figure P3.22(a)–(f).
(b) $x(t)$ periodic with period 2 and

$$x(t) = e^{-t} \quad \text{for} \quad -1 < t < 1$$

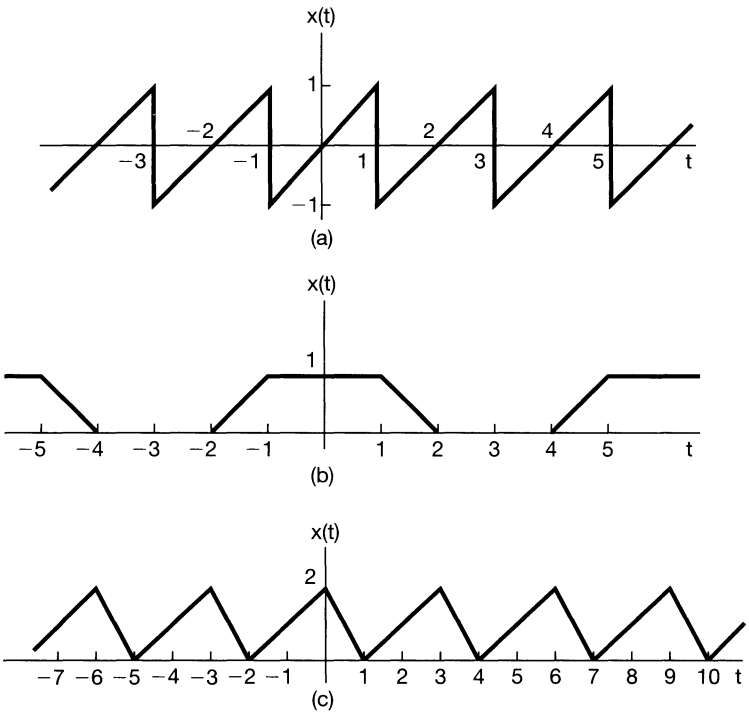


Figure P3.22

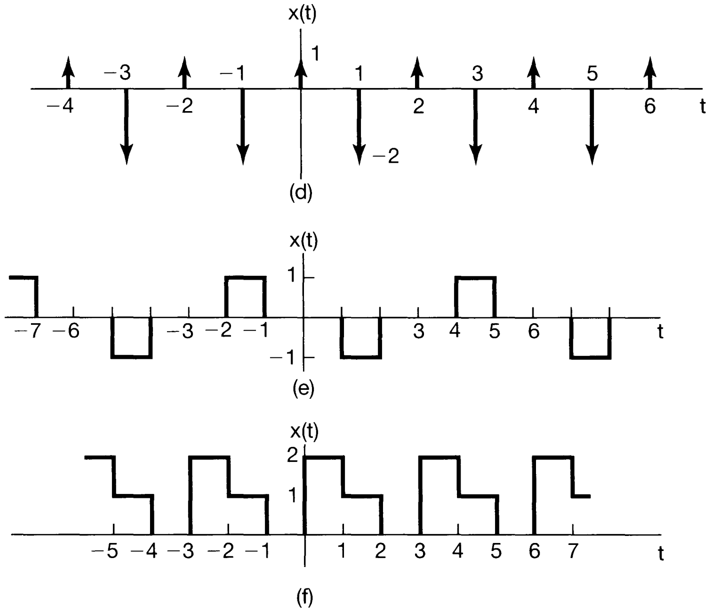


Figure P3.22 *Continued*

(c) $x(t)$ periodic with period 4 and

$$x(t) = \begin{cases} \sin \pi t, & 0 \le t \le 2 \\ 0, & 2 < t \le 4 \end{cases}$$

3.23. In each of the following, we specify the Fourier series coefficients of a continuous-time signal that is periodic with period 4. Determine the signal $x(t)$ in each case.

(a) $a_k = \begin{cases} 0, & k = 0 \\ (j)^k \frac{\sin k\pi/4}{k\pi}, & \text{otherwise} \end{cases}$

(b) $a_k = (-1)^k \frac{\sin k\pi/8}{2k\pi}, \quad a_0 = \frac{1}{16}$

(c) $a_k = \begin{cases} jk, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$

(d) $a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$

3.24. Let

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \end{cases}$$

be a periodic signal with fundamental period $T = 2$ and Fourier coefficients a_k.

(a) Determine the value of a_0.

(b) Determine the Fourier series representation of $dx(t)/dt$.

(c) Use the result of part (b) and the differentiation property of the continuous-time Fourier series to help determine the Fourier series coefficients of $x(t)$.

3.25. Consider the following three continuous-time signals with a fundamental period of $T = 1/2$:

$$\begin{aligned} x(t) &= \cos(4\pi t), \\ y(t) &= \sin(4\pi t), \\ z(t) &= x(t)y(t). \end{aligned}$$

(a) Determine the Fourier series coefficients of $x(t)$.
(b) Determine the Fourier series coefficients of $y(t)$.
(c) Use the results of parts (a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.
(d) Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part (c).

3.26. Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$a_k = \begin{cases} 2, & k = 0 \\ j(\frac{1}{2})^{|k|}, & \text{otherwise} \end{cases}.$$

Use Fourier series properties to answer the following questions:
(a) Is $x(t)$ real?
(b) Is $x(t)$ even?
(c) Is $dx(t)/dt$ even?

3.27. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N = 5$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 2, a_2 = a_{-2}^* = 2e^{j\pi/6}, \quad a_4 = a_{-4}^* = e^{j\frac{\pi}{3}}.$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(\omega_k n + \phi_k).$$

3.28. Determine the Fourier series coefficients for each of the following discrete-time periodic signals. Plot the magnitude and phase of each set of coefficients a_k.
(a) Each $x[n]$ depicted in Figure P3.28(a)–(c)
(b) $x[n] = \sin(2\pi n/3)\cos(\pi n/2)$
(c) $x[n]$ periodic with period 4 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \le n \le 3$$

(d) $x[n]$ periodic with period 12 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \le n \le 11$$

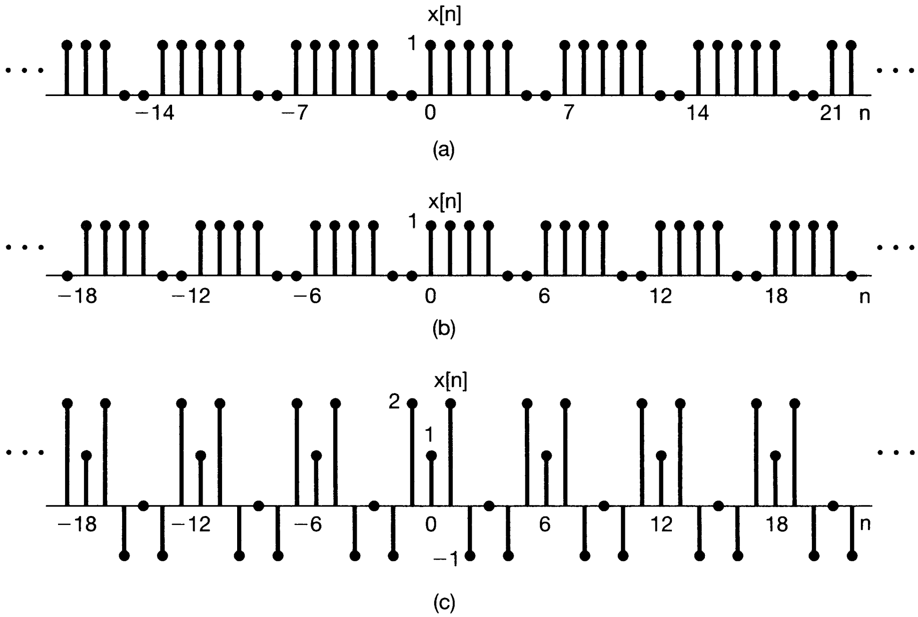


Figure P3.28

3.29. In each of the following, we specify the Fourier series coefficients of a signal that is periodic with period 8. Determine the signal $x[n]$ in each case.

(a) $a_k = \cos\left(\frac{k\pi}{4}\right) + \sin\left(\frac{3k\pi}{4}\right)$ **(b)** $a_k = \begin{cases} \sin\left(\frac{k\pi}{3}\right), & 0 \le k \le 6 \\ 0, & k = 7 \end{cases}$

(c) a_k as in Figure P3.29(a) **(d)** a_k as in Figure P3.29(b)

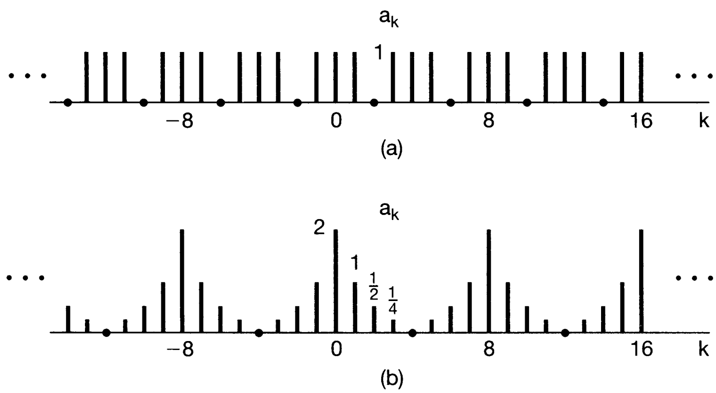


Figure P3.29

3.30. Consider the following three discrete-time signals with a fundamental period of 6:

$$x[n] = 1 + \cos\left(\frac{2\pi}{6}n\right), \qquad y[n] = \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right), \qquad z[n] = x[n]y[n].$$

(a) Determine the Fourier series coefficients of $x[n]$.
(b) Determine the Fourier series coefficients of $y[n]$.
(c) Use the results of parts (a) and (b), along with the multiplication property of the discrete-time Fourier series, to determine the Fourier series coefficients of $z[n] = x[n]y[n]$.
(d) Determine the Fourier series coefficients of $z[n]$ through direct evaluation, and compare your result with that of part (c).

3.31. Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & 8 \leq n \leq 9 \end{cases}$$

be a periodic signal with fundamental period $N = 10$ and Fourier series coefficients a_k. Also, let

$$g[n] = x[n] - x[n-1].$$

(a) Show that $g[n]$ has a fundamental period of 10.
(b) Determine the Fourier series coefficients of $g[n]$.
(c) Using the Fourier series coefficients of $g[n]$ and the First-Difference property in Table 3.2, determine a_k for $k \neq 0$.

3.32. Consider the signal $x[n]$ depicted in Figure P3.32. This signal is periodic with period $N = 4$. The signal can be expressed in terms of a discrete-time Fourier series as

$$x[n] = \sum_{k=0}^{3} a_k e^{jk(2\pi/4)n}. \quad \text{(P3.32–1)}$$

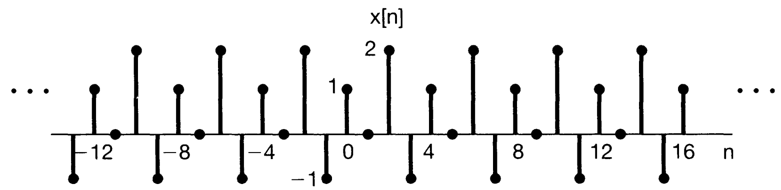


Figure P3.32

As mentioned in the text, one way to determine the Fourier series coefficients is to treat eq. (P3.32–1) as a set of four linear equations (for $n = 0, 1, 2, 3$) in four unknowns (a_0, a_1, a_2, and a_3).

(a) Write out these four equations explicitly, and solve them directly using any standard technique for solving four equations in four unknowns. (Be sure first to reduce the foregoing complex exponentials to the simplest form.)
(b) Check your answer by calculating the a_k directly, using the discrete-time Fourier series analysis equation

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n}.$$

3.33. Consider a causal continuous-time LTI system whose input $x(t)$ and output $y(t)$ are related by the following differential equation:

$$\frac{d}{dt}y(t) + 4y(t) = x(t).$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:
(a) $x(t) = \cos 2\pi t$
(b) $x(t) = \sin 4\pi t + \cos(6\pi t + \pi/4)$

3.34. Consider a continuous-time LTI system with impulse response

$$h(t) = e^{-4|t|}.$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:
(a) $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$
(b) $x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \delta(t-n)$
(c) $x(t)$ is the periodic wave depicted in Figure P3.34.

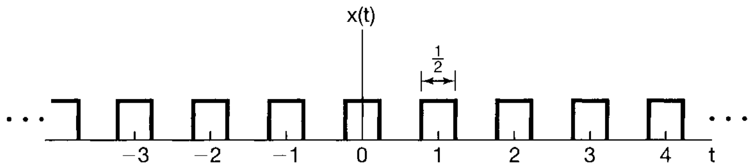


Figure P3.34

3.35. Consider a continuous-time LTI system S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \geq 250 \\ 0, & \text{otherwise} \end{cases}.$$

When the input to this system is a signal $x(t)$ with fundamental period $T = \pi/7$ and Fourier series coefficients a_k, it is found that the output $y(t)$ is identical to $x(t)$. For what values of k is it guaranteed that $a_k = 0$?

3.36. Consider a causal discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs:
(a) $x[n] = \sin(\frac{3\pi}{4}n)$
(b) $x[n] = \cos(\frac{\pi}{4}n) + 2\cos(\frac{\pi}{2}n)$

3.37. Consider a discrete-time LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{|n|}.$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs:

(a) $x[n] = \sum_{k=-\infty}^{\infty}\delta[n-4k]$

(b) $x[n]$ is periodic with period 6 and

$$x[n] = \begin{cases} 1, & n = 0, \pm 1 \\ 0, & n = \pm 2, \pm 3 \end{cases}$$

3.38. Consider a discrete-time LTI system with impulse response

$$h[n] = \begin{cases} 1, & 0 \le n \le 2 \\ -1, & -2 \le n \le -1 \\ 0, & \text{otherwise} \end{cases}.$$

Given that the input to this system is

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-4k],$$

determine the Fourier series coefficients of the output $y[n]$.

3.39. Consider a discrete-time LTI system S whose frequency response is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |\omega| < \pi \end{cases}.$$

Show that if the input $x[n]$ to this system has a period $N = 3$, the output $y[n]$ has only one nonzero Fourier series coefficient per period.

ADVANCED PROBLEMS

3.40. Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of each of the following signals in terms of a_k:

(a) $x(t - t_0) + x(t + t_0)$

(b) $\mathcal{E}v\{x(t)\}$

(c) $\mathcal{R}e\{x(t)\}$

(d) $\frac{d^2x(t)}{dt^2}$

(e) $x(3t - 1)$ [for this part, first determine the period of $x(3t - 1)$]

3.41. Suppose we are given the following information about a continuous-time periodic signal with period 3 and Fourier coefficients a_k:

1. $a_k = a_{k+2}$.
2. $a_k = a_{-k}$.
3. $\int_{-0.5}^{0.5} x(t)dt = 1$.
4. $\int_1^2 x(t)dt = 2$.

Determine $x(t)$.

3.42. Let $x(t)$ be a real-valued signal with fundamental period T and Fourier series coefficients a_k.

(a) Show that $a_k = a_{-k}^*$ and a_0 must be real.

(b) Show that if $x(t)$ is even, then its Fourier series coefficients must be real and even.

(c) Show that if $x(t)$ is odd, then its Fourier series coefficients are imaginary and odd and $a_0 = 0$.

(d) Show that the Fourier coefficients of the even part of $x(t)$ are equal to $\mathcal{R}e\{a_k\}$.

(e) Show that the Fourier coefficients of the odd part of $x(t)$ are equal to $j\mathcal{I}m\{a_k\}$.

3.43. **(a)** A continuous-time periodic signal $x(t)$ with period T is said to be *odd harmonic* if, in its Fourier series representation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \qquad \text{(P3.43–1)}$$

$a_k = 0$ for every non-zero even integer k.

(i) Show that if $x(t)$ is odd harmonic, then

$$x(t) = -x\left(t + \frac{T}{2}\right). \qquad \text{(P3.43–2)}$$

(ii) Show that if $x(t)$ satisfies eq. (P3.43–2), then it is odd harmonic.

(b) Suppose that $x(t)$ is an odd-harmonic periodic signal with period 2 such that

$$x(t) = t \quad \text{for } 0 < t < 1.$$

Sketch $x(t)$ and find its Fourier series coefficients.

(c) Analogously, to an odd-harmonic signal, we could define an even-harmonic signal as a signal for which $a_k = 0$ for k odd in the representation in eq. (P3.43–1). Could T be the fundamental period for such a signal? Explain your answer.

(d) More generally, show that T is the fundamental period of $x(t)$ in eq. (P3.43–1) if one of two things happens:

(1) Either a_1 or a_{-1} is nonzero;

or

(2) There are two integers k and l that have no common factors and are such that both a_k and a_l are nonzero.

3.44. Suppose we are given the following information about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 6$ and has Fourier coefficients a_k.
3. $a_k = 0$ for $k = 0$ and $k > 2$.
4. $x(t) = -x(t-3)$.
5. $\frac{1}{6}\int_{-3}^{3}|x(t)|^2\,dt = \frac{1}{2}$.
6. a_1 is a positive real number.

Show that $x(t) = A\cos(Bt + C)$, and determine the values of the constants A, B, and C.

3.45. Let $x(t)$ be a real periodic signal with Fourier series representation given in the sine-cosine form of eq. (3.32); i.e.,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]. \qquad \text{(P3.45–1)}$$

(a) Find the exponential Fourier series representation of the even and odd parts of $x(t)$; that is, find the coefficients α_k and β_k in terms of the coefficients in eq. (P3.45–1) so that

$$\mathcal{E}v\{x(t)\} = \sum_{k=-\infty}^{+\infty} \alpha_k e^{jk\omega_0 t},$$

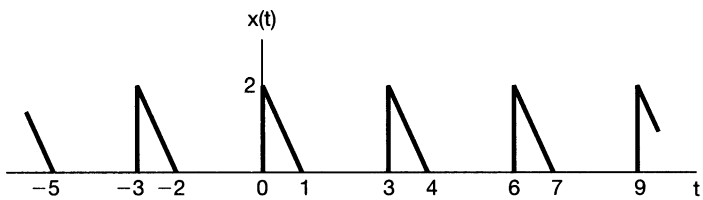
$$\mathcal{O}d\{x(t)\} = \sum_{k=-\infty}^{+\infty} \beta_k e^{jk\omega_0 t}.$$

(b) What is the relationship between α_k and α_{-k} in part (a)? What is the relationship between β_k and β_{-k}?

(c) Suppose that the signals $x(t)$ and $z(t)$ shown in Figure P3.45 have the sine-cosine series representations

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos\left(\frac{2\pi kt}{3}\right) - C_k \sin\left(\frac{2\pi kt}{3}\right)\right],$$

$$z(t) = d_0 + 2\sum_{k=1}^{\infty}\left[E_k \cos\left(\frac{2\pi kt}{3}\right) - F_k \sin\left(\frac{2\pi kt}{3}\right)\right].$$



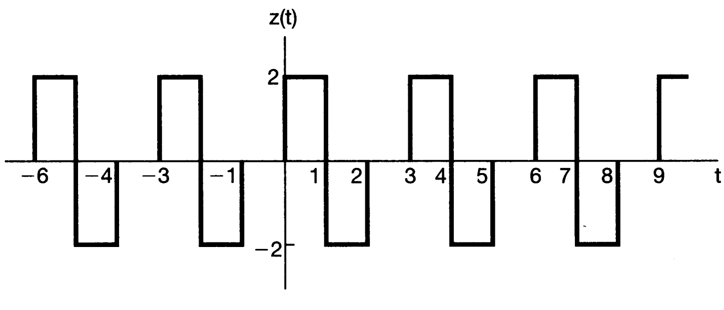


Figure P3.45

Sketch the signal

$$y(t) = 4(a_0 + d_0) + 2\sum_{k=1}^{\infty}\left\{\left[B_k + \frac{1}{2}E_k\right]\cos\left(\frac{2\pi kt}{3}\right) + F_k \sin\left(\frac{2\pi kt}{3}\right)\right\}.$$

3.46 In this problem, we derive two important properties of the continuous-time Fourier series: the multiplication property and Parseval's relation. Let $x(t)$ and $y(t)$ both be continuous-time periodic signals having period T_0 and with Fourier series representations given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad y(t) = \sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_0 t}. \tag{P3.46–1}$$

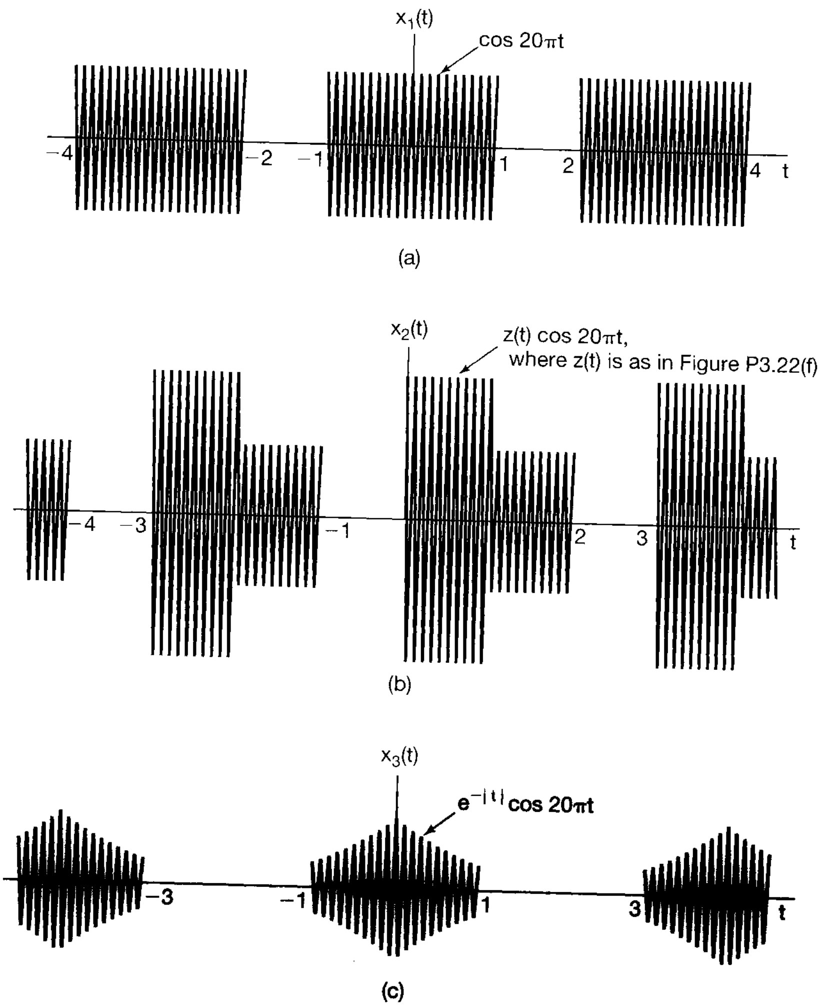


Figure P3.46

(a) Show that the Fourier series coefficients of the signal

$$z(t) = x(t)y(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

are given by the discrete convolution

$$c_k = \sum_{n=-\infty}^{+\infty} a_n b_{k-n}.$$

(b) Use the result of part (a) to compute the Fourier series coefficients of the signals $x_1(t)$, $x_2(t)$, and $x_3(t)$ depicted in Figure P3.46.

(c) Suppose that $y(t)$ in eq. (P3.46–1) equals $x^*(t)$. Express the b_k in the equation in terms of a_k, and use the result of part (a) to prove Parseval's relation for periodic signals—that is,

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2.$$

3.47 Consider the signal

$$x(t) = \cos 2\pi t.$$

Since $x(t)$ is periodic with a fundamental period of 1, it is also periodic with a period of N, where N is any positive integer. What are the Fourier series coefficients of $x(t)$ if we regard it as a periodic signal with period 3?

3.48. Let $x[n]$ be a periodic sequence with period N and Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \qquad \text{(P3.48–1)}$$

The Fourier series coefficients for each of the following signals can be expressed in terms of a_k in eq. (P3.48–1). Derive the expressions.

(a) $x[n-n_0]$

(b) $x[n]-x[n-1]$

(c) $x[n]-x[n-\frac{N}{2}]$ (assume that N is even)

(d) $x[n]+x[n+\frac{N}{2}]$ (assume that N is even; note that this signal is periodic with period $N/2$)

(e) $x^*[-n]$

(f) $(-1)^n x[n]$ (assume that N is even)

(g) $(-1)^n x[n]$ (assume that N is odd; note that this signal is periodic with period $2N$)

(h) $y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

3.49. Let $x[n]$ be a periodic sequence with period N and Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \qquad \text{(P3.49–1)}$$

(a) Suppose that N is even and that $x[n]$ in eq. (P3.49–1) satisfies

$$x[n] = -x\left[n + \frac{N}{2}\right] \text{ for all } n.$$

Show that $a_k = 0$ for all even integers k.

(b) Suppose that N is divisible by 4. Show that if

$$x[n] = -x\left[n + \frac{N}{4}\right] \text{ for all } n,$$

then $a_k = 0$ for every value of k that is a multiple of 4.

(c) More generally, suppose that N is divisible by an integer M. Show that if

$$\sum_{r=0}^{(N/M)-1} x\left[n + r\frac{N}{M}\right] = 0 \text{ for all } n,$$

then $a_k = 0$ for every value of k that is a multiple of M.

3.50. Suppose we are given the following information about a periodic signal $x[n]$ with period 8 and Fourier coefficients a_k:

1. $a_k = -a_{k-4}$.
2. $x[2n + 1] = (-1)^n$.

Sketch one period of $x[n]$.

3.51. Let $x[n]$ be a periodic signal with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$. A signal

$$y[n] = \left(\frac{1 + (-1)^n}{2}\right)x[n - 1]$$

with period $N = 8$ is generated. Denoting the Fourier series coefficients of $y[n]$ by b_k, find a function $f[k]$ such that

$$b_k = f[k]a_k.$$

3.52. $x[n]$ is a real periodic signal with period N and complex Fourier series coefficients a_k. Let the Cartesian form for a_k be denoted by

$$a_k = b_k + jc_k,$$

where b_k and c_k are both real.

(a) Show that $a_{-k} = a_k^*$. What is the relation between b_k and b_{-k}? What is the relation between c_k and c_{-k}?

(b) Suppose that N is even. Show that $a_{N/2}$ is real.

(c) Show that $x[n]$ can also be expressed as a trigonometric Fourier series of the form

$$x[n] = a_0 + 2\sum_{k=1}^{(N-1)/2} b_k \cos\left(\frac{2\pi kn}{N}\right) - c_k \sin\left(\frac{2\pi kn}{N}\right)$$

if N is odd or as

$$x[n] = (a_0 + a_{N/2}(-1)^n) + 2\sum_{k=1}^{(N-2)/2} b_k \cos\left(\frac{2\pi kn}{N}\right) - c_k \sin\left(\frac{2\pi kn}{N}\right)$$

if N is even.

(d) Show that if the polar form of a_k is $A_k e^{j\theta k}$, then the Fourier series representation for $x[n]$ can also be written as

$$x[n] = a_0 + 2\sum_{k=1}^{(N-1)/2} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)$$

if N is odd or as

$$x[n] = (a_0 + a_{N/2}(-1)^n) + 2\sum_{k=1}^{(N/2)-1} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)$$

if N is even.

(e) Suppose that $x[n]$ and $z[n]$, as depicted in Figure P3.52, have the sine-cosine series representations

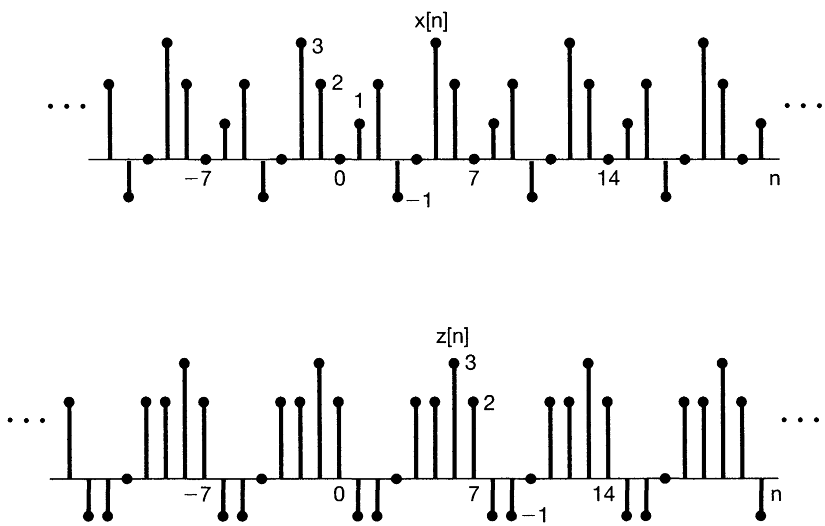


Figure P3.52

$$x[n] = a_0 + 2\sum_{k=1}^{3}\left\{b_k \cos\left(\frac{2\pi kn}{7}\right) - c_k \sin\left(\frac{2\pi kn}{7}\right)\right\},$$

$$z[n] = d_0 + 2\sum_{k=1}^{3}\left\{d_k \cos\left(\frac{2\pi kn}{7}\right) - f_k \sin\left(\frac{2\pi kn}{7}\right)\right\}.$$

Sketch the signal

$$y[n] = a_0 - d_0 + 2\sum_{k=1}^{3}\left\{d_k \cos\left(\frac{2\pi kn}{7}\right) + (f_k - c_k) \sin\left(\frac{2\pi kn}{7}\right)\right\}.$$

3.53. Let $x[n]$ be a real periodic signal with period N and Fourier coefficients a_k.

(a) Show that if N is even, at least two of the Fourier coefficients within one period of a_k are real.

(b) Show that if N is odd, at least one of the Fourier coefficients within one period of a_k is real.

3.54. Consider the function

$$a[k] = \sum_{n=0}^{N-1} e^{j(2\pi/N)kn}.$$

(a) Show that $a[k] = N$ for $k = 0, \pm N, \pm 2N, \pm 3N, \ldots$.

(b) Show that $a[k] = 0$ whenever k is not an integer multiple of N. (*Hint:* Use the finite sum formula.)

(c) Repeat parts (a) and (b) if

$$a[k] = \sum_{n=<N>} e^{j(2\pi/N)kn}.$$

3.55. Let $x[n]$ be a periodic signal with fundamental period N and Fourier series coefficients a_k. In this problem, we derive the time-scaling property

$$x_{(m)}[n] = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \pm 2m, \cdots \\ 0, & \text{elsewhere} \end{cases}$$

listed in Table 3.2.

(a) Show that $x_{(m)}[n]$ has period of mN.

(b) Show that if

$$x[n] = v[n] + w[n],$$

then

$$x_{(m)}[n] = v_{(m)}[n] + w_{(m)}[n].$$

(c) Assuming that $x[n] = e^{j2\pi k_0 n/N}$ for some integer k_0, verify that

$$x_{(m)}[n] = \frac{1}{m}\sum_{l=0}^{m-1} e^{j2\pi(k_0+lN)n/(mN)}.$$

That is, one complex exponential in $x[n]$ becomes a linear combination of m complex exponentials in $x_{(m)}[n]$.

(d) Using the results of parts (a), (b), and (c), show that if $x[n]$ has the Fourier coefficients a_k, then $x_{(m)}[n]$ must have the Fourier coefficients $\frac{1}{m}a_k$.

3.56. Let $x[n]$ be a periodic signal with period N and Fourier coefficients a_k.

(a) Express the Fourier coefficients b_k of $|x[n]|^2$ in terms of a_k.

(b) If the coefficients a_k are real, is it guaranteed that the coefficients b_k are also real?

3.57. (a) Let

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n} \tag{P3.57–1}$$

and

$$y[n] = \sum_{k=0}^{N-1} b_k e^{jk(2\pi/N)n}$$

be periodic signals. Show that

$$x[n]y[n] = \sum_{k=0}^{N-1} c_k e^{jk(2\pi/N)n},$$

where

$$c_k = \sum_{l=0}^{N-1} a_l b_{k-l} = \sum_{l=0}^{N-1} a_{k-l} b_l.$$

(b) Generalize the result of part (a) by showing that

$$c_k = \sum_{l=<N>} a_l b_{k-l} = \sum_{l=<N>} a_{k-l} b_l.$$

(c) Use the result of part (b) to find the Fourier series representation of the following signals, where $x[n]$ is given by eq. (P3.57–1).

(i) $x[n]\cos\left(\frac{6\pi n}{N}\right)$

(ii) $x[n]\sum_{r=-\infty}^{+\infty}\delta[n-rN]$

(iii) $x[n]\left(\sum_{r=-\infty}^{+\infty}\delta\left[n-\frac{rN}{3}\right]\right)$ (assume that N is divisible by 3)

(d) Find the Fourier series representation for the signal $x[n]y[n]$, where

$$x[n] = \cos(\pi n/3)$$

and

$$y[n] = \begin{cases} 1, & |n| \leq 3 \\ 0, & 4 \leq |n| \leq 6 \end{cases}$$

is periodic with period 12.

(e) Use the result of part (b) to show that

$$\sum_{n=<N>} x[n]y[n] = N \sum_{l=<N>} a_l b_{-l},$$

and from this expression, derive Parseval's relation for discrete-time periodic signals.

3.58. Let $x[n]$ and $y[n]$ be periodic signals with common period N, and let

$$z[n] = \sum_{r=<N>} x[r]y[n-r]$$

be their periodic convolution.

(a) Show that $z[n]$ is also periodic with period N.

(b) Verify that if a_k, b_k, and c_k are the Fourier coefficients of $x[n]$, $y[n]$, and $z[n]$, respectively, then

$$c_k = N a_k b_k.$$

(c) Let

$$x[n] = \sin\left(\frac{3\pi n}{4}\right)$$

and

$$y[n] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases}$$

be two signals that are periodic with period 8. Find the Fourier series representation for the periodic convolution of these signals.

(d) Repeat part (c) for the following two periodic signals that also have period 8:

$$x[n] = \begin{cases} \sin\left(\frac{3\pi n}{4}\right), & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases},$$

$$y[n] = \left(\frac{1}{2}\right)^n, 0 \leq n \leq 7.$$

3.59. **(a)** Suppose $x[n]$ is a periodic signal with period N. Show that the Fourier series coefficients of the periodic signal

$$g(t) = \sum_{k=-\infty}^{\infty} x[k]\,\delta(t-kT)$$

are periodic with period N.

(b) Suppose that $x(t)$ is a periodic signal with period T and Fourier series coefficients a_k with period N. Show that there must exist a periodic sequence $g[n]$ such that

$$x(t) = \sum_{k=-\infty}^{\infty} g[k]\,\delta(t - kT/N).$$

(c) Can a continuous periodic signal have periodic Fourier coefficients?

3.60. Consider the following pairs of signals $x[n]$ and $y[n]$. For each pair, determine whether there is a discrete-time LTI system for which $y[n]$ is the output when the corresponding $x[n]$ is the input. If such a system exists, determine whether the system is unique (i.e., whether there is more than one LTI system with the given input-output pair). Also, determine the frequency response of an LTI system with the desired behavior. If no such LTI system exists for a given $x[n]$, $y[n]$ pair, explain why.

(a) $x[n] = (\frac{1}{2}^n)$, $y[n] = (\frac{1}{4}^n)$
(b) $x[n] = (\frac{1}{2}^n)u[n]$, $y[n] = (\frac{1}{4})^n u[n]$
(c) $x[n] = (\frac{1}{2}^n)u[n]$, $y[n] = 4^n u[-n]$
(d) $x[n] = e^{jn/8}$, $y[n] = 2e^{jn/8}$
(e) $x[n] = e^{jn/8}u[n]$, $y[n] = 2e^{jn/8}u[n]$
(f) $x[n] = j^n$, $y[n] = 2j^n(1 - j)$
(g) $x[n] = \cos(\pi n/3), y[n] = \cos(\pi n/3) + \sqrt{3}\sin(\pi n/3)$
(h) $x[n]$ and $y_1[n]$ as in Figure P3.60
(i) $x[n]$ and $y_2[n]$ as in Figure P3.60

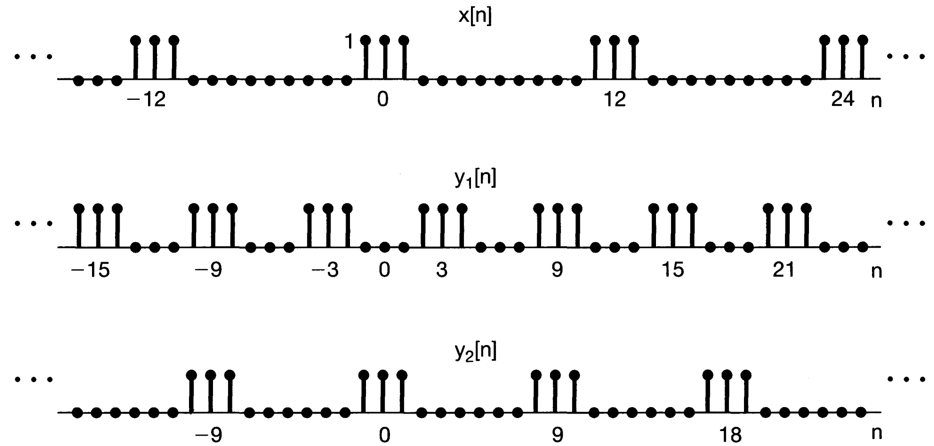


Figure P3.60

3.61. As we have seen, the techniques of Fourier analysis are of value in examining continuous-time LTI systems because periodic complex exponentials are eigenfunctions for LTI systems. In this problem, we wish to substantiate the following statement: Although some LTI systems may have additional eigenfunctions, the complex exponentials are the *only* signals that are eigenfunctions of *every* LTI system.

(a) What are the eigenfunctions of the LTI system with unit impulse response $h(t) = \delta(t)$? What are the associated eigenvalues?

(b) Consider the LTI system with unit impulse response $h(t) = \delta(t - T)$. Find a signal that is not of the form e^{st}, but that is an eigenfunction of the system with eigenvalue 1. Similarly, find the eigenfunctions with eigenvalues 1/2 and 2 that are not complex exponentials. (*Hint:* You can find impulse trains that meet these requirements.)

(c) Consider a stable LTI system with impulse response $h(t)$ that is real and even. Show that $\cos \omega t$ and $\sin \omega t$ are eigenfunctions of this system.

(d) Consider the LTI system with impulse response $h(t) = u(t)$. Suppose that $\phi(t)$ is an eigenfunction of this system with eigenvalue λ. Find the differential equation that $\phi(t)$ must satisfy, and solve the equation. This result, together with those of parts (a) through (c), should prove the validity of the statement made at the beginning of the problem.

3.62. One technique for building a dc power supply is to take an ac signal and full-wave rectify it. That is, we put the ac signal $x(t)$ through a system that produces $y(t) = |x(t)|$ as its output.

(a) Sketch the input and output waveforms if $x(t) = \cos t$. What are the fundamental periods of the input and output?

(b) If $x(t) = \cos t$, determine the coefficients of the Fourier series for the output $y(t)$.

(c) What is the amplitude of the dc component of the input signal? What is the amplitude of the dc component of the output signal?

3.63. Suppose that a continuous-time periodic signal is the input to an LTI system. The signal has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{jk(\pi/4)t},$$

where α is a real number between 0 and 1, and the frequency response of the system is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq W \\ 0, & |\omega| > W \end{cases}.$$

How large must W be in order for the output of the system to have at least 90% of the average energy per period of $x(t)$?

3.64. As we have seen in this chapter, the concept of an eigenfunction is an extremely important tool in the study of LTI systems. The same can be said for linear, but time-varying, systems. Specifically, consider such a system with input $x(t)$ and output $y(t)$. We say that a signal $\phi(t)$ is an *eigenfunction* of the system if

$$\phi(t) \longrightarrow \lambda\phi(t).$$

That is, if $x(t) = \phi(t)$, then $y(t) = \lambda\phi(t)$, where the complex constant λ is called the *eigenvalue associated with* $\phi(t)$.

(a) Suppose that we can represent the input $x(t)$ to our system as a linear combination of eigenfunctions $\phi_k(t)$, each of which has a corresponding eigenvalue λ_k; that is,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \phi_k(t).$$

Express the output $y(t)$ of the system in terms of $\{c_k\}$, $\{\phi_k(t)\}$, and $\{\lambda_k\}$.

(b) Consider the system characterized by the differential equation

$$y(t) = t^2 \frac{d^2 x(t)}{dt^2} + t\frac{dx(t)}{dt}.$$

Is this system linear? Is it time invariant?

(c) Show that the functions

$$\phi_k(t) = t^k$$

are eigenfunctions of the system in part (b). For each $\phi_k(t)$, determine the corresponding eigenvalue λ_k.

(d) Determine the output of the system if

$$x(t) = 10t^{-10} + 3t + \frac{1}{2}t^4 + \pi.$$

EXTENSION PROBLEMS

3.65. Two functions $u(t)$ and $v(t)$ are said to be *orthogonal over the interval (a,b)* if

$$\int_a^b u(t)v^*(t)\,dt = 0. \qquad \text{(P3.65–1)}$$

If, in addition,

$$\int_a^b |u(t)|^2\,dt = 1 = \int_a^b |v(t)|^2\,dt,$$

the functions are said to be *normalized* and hence are called *orthonormal.* A set of functions $\{\phi_k(t)\}$ is called an *orthogonal (orthonormal) set* if each pair of functions in the set is orthogonal (orthonormal).

(a) Consider the pairs of signals $u(t)$ and $v(t)$ depicted in Figure P3.65. Determine whether each pair is orthogonal over the interval (0, 4).

(b) Are the functions $\sin m\omega_0 t$ and $\sin n\omega_0 t$ orthogonal over the interval $(0, T)$, where $T = 2\pi/\omega_0$? Are they also orthonormal?

(c) Repeat part (b) for the functions $\phi_m(t)$ and $\phi_n(t)$, where

$$\phi_k(t) = \frac{1}{\sqrt{T}}[\cos k\omega_0 t + \sin k\omega_0 t].$$

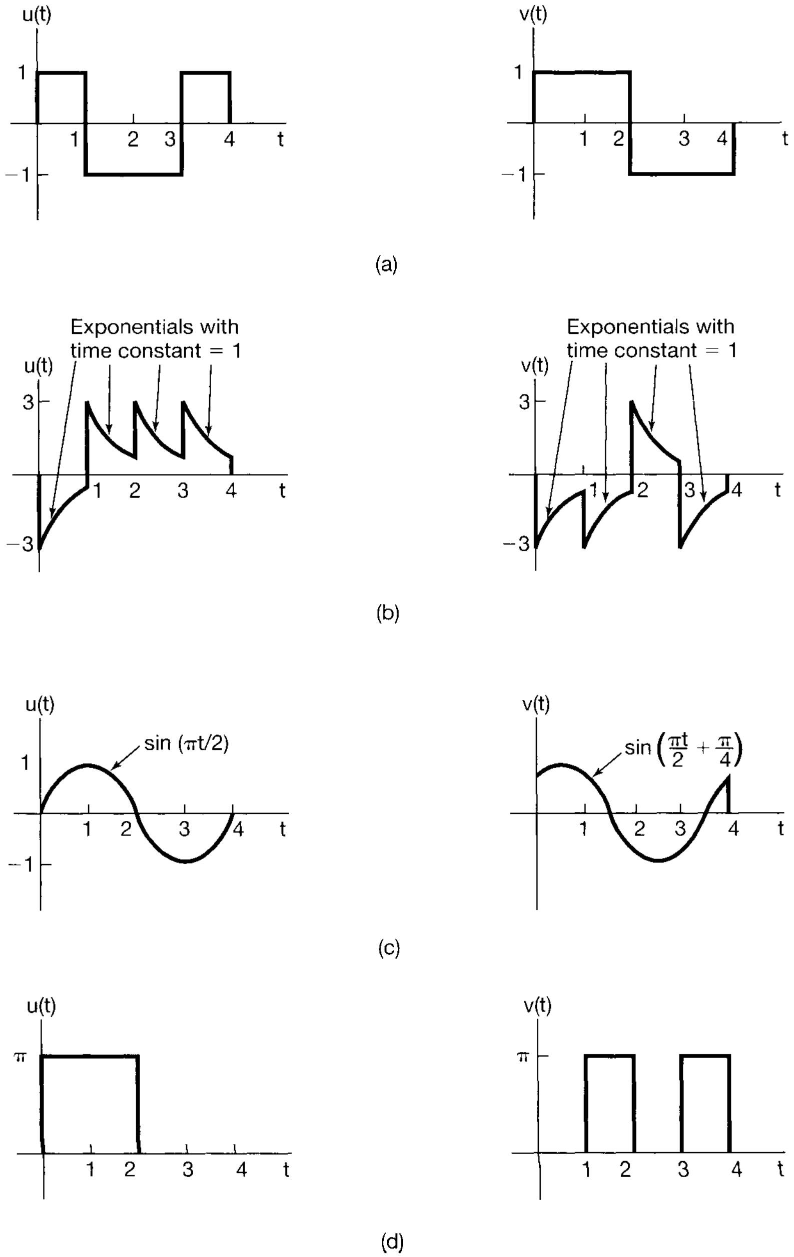


Figure P3.65

(d) Show that the functions $\phi_k(t) = e^{jk\omega_0 t}$ are orthogonal over *any* interval of length $T = 2\pi/\omega_0$. Are they orthonormal?

(e) Let $x(t)$ be an arbitrary signal, and let $x_o(t)$ and $x_e(t)$ be, respectively, the odd and even parts of $x(t)$. Show that $x_o(t)$ and $x_e(t)$ are orthogonal over the interval $(-T, T)$ for any T.

(f) Show that if $\{\phi_k(t)\}$ is a set of orthogonal signals over the interval (a, b), then the set $\{(1/\sqrt{A_k})\phi_k(t)\}$, where

$$A_k = \int_a^b |\phi_k(t)|^2\,dt,$$

is orthonormal.

(g) Let $\{\phi_i(t)\}$ be a set of orthonormal signals on the interval (a, b), and consider a signal of the form

$$x(t) = \sum_i a_i \phi_i(t),$$

where the a_i are complex constants. Show that

$$\int_a^b |x(t)|^2\,dt = \sum_i |a_i|^2.$$

(h) Suppose that $\phi_1(t), \ldots, \phi_N(t)$ are nonzero only in the time interval $0 \le t \le T$ and that they are orthonormal over this time interval. Let L_i denote the LTI system with impulse response

$$h_i(t) = \phi_i(T - t). \tag{P3.65–2}$$

Show that if $\phi_j(t)$ is applied to this system, then the output at time T is 1 if $i = j$ and 0 if $i \neq j$. The system with impulse response given by eq. (P3.65–2) was referred to in Problems 2.66 and 2.67 as the *matched filter* for the signal $\phi_i(t)$.

3.66. The purpose of this problem is to show that the representation of an arbitrary periodic signal by a Fourier series or, more generally, as a linear combination of any set of orthogonal functions is computationally efficient and in fact very useful for obtaining good approximations of signals.[12]

Specifically, let $\{\phi_i(t)\}$, $i = 0, \pm 1, \pm 2, \ldots$ be a set of orthonormal functions on the interval $a \le t \le b$, and let $x(t)$ be a given signal. Consider the following approximation of $x(t)$ over the interval $a \le t \le b$:

$$\hat{x}_n(t) = \sum_{i=-N}^{+N} a_i \phi_i(t). \tag{P3.66–1}$$

Here, the a_i are (in general, complex) constants. To measure the deviation between $x(t)$ and the series approximation $\hat{x}_N(t)$, we consider the error $e_N(t)$ defined as

$$e_N(t) = x(t) - \hat{x}_N(t). \tag{P3.66–2}$$

A reasonable and widely used criterion for measuring the quality of the approximation is the energy in the error signal over the interval of interest—that is, the integral

[12] See Problem 3.65 for the definitions of orthogonal and orthonormal functions.

of the square of the magnitude of the error over the interval $a \leq t \leq b$:

$$E = \int_a^b |e_N(t)|^2 \, dt. \tag{P3.66–3}$$

(a) Show that E is minimized by choosing

$$a_i = \int_a^b x(t)\phi_i^*(t) \, dt. \tag{P3.66–4}$$

[*Hint:* Use eqs. (P3.66–1)–(P3.66–3) to express E in terms of a_i, $\phi_i(t)$, and $x(t)$. Then express a_i in rectangular coordinates as $a_i = b_i + jc_i$, and show that the equations

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0, i = 0, \pm 1, \pm 2, \ldots, N$$

are satisfied by the a_i as given by eq. (P3.66–4).]

(b) How does the result of part (a) change if

$$A_i = \int_a^b |\phi_i(t)|^2 \, dt$$

and the $\{\phi_i(t)\}$ are orthogonal but not orthonormal?

(c) Let $\phi_n(t) = e^{jn\omega_0 t}$, and choose any interval of length $T_0 = 2\pi/\omega_0$. Show that the a_i that minimize E are as given in eq. (3.50).

(d) The set of *Walsh functions* is an often-used set of orthonormal functions. (See Problem 2.66.) The set of five Walsh functions, $\phi_0(t), \phi_1(t), \ldots, \phi_4(t)$, is illustrated in Figure P3.66, where we have scaled time so that the $\phi_i(t)$ are nonzero and orthonormal over the interval $0 \leq t \leq 1$. Let $x(t) = \sin \pi t$. Find the approximation of $x(t)$ of the form

$$\hat{x}(t) = \sum_{i=0}^{4} a_i \phi_i(t)$$

such that

$$\int_0^1 |x(t) - \hat{x}(t)|^2 \, dt$$

is minimized.

(e) Show that $\hat{x}_N(t)$ in eq. (P3.66–1) and $e_N(t)$ in eq. (P3.66–2) are orthogonal if the a_i are chosen as in eq. (P3.66–4).

The results of parts (a) and (b) are extremely important in that they show that each coefficient a_i is *independent* of all the other a_j's, $i \neq j$. Thus, if we add more terms to the approximation [e.g., if we compute the approximation $\hat{x}_{N+1}(t)$], the coefficients of $\phi_i(t)$, $i = 1, \ldots, N$, that were previously determined will not change. In contrast to this, consider another type of se-

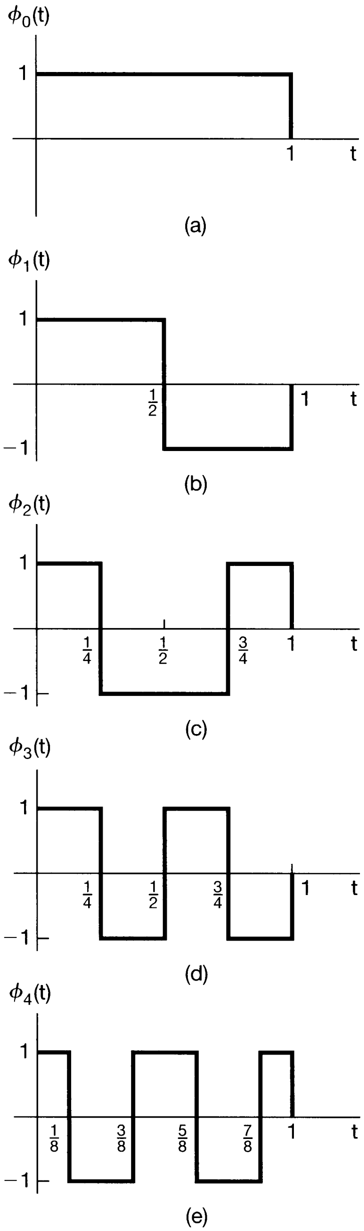


Figure P3.66

ries expansion, the polynomial Taylor series. The *infinite* Taylor series for e^t is $e^t = 1 + t + t^2/2! + \ldots$, but as we shall show, when we consider a *finite* polynomial series and the error criterion of eq. (P3.66–3), we get a very different result.

Specifically, let $\phi_0(t) = 1$, $\phi_1(t) = t$, $\phi_2(t) = t^2$, and so on.

(f) Are the $\phi_i(t)$ orthogonal over the interval $0 \leq t \leq 1$?

(g) Consider an approximation of $x(t) = e^t$ over the interval $0 \leq t \leq 1$ of the form

$$\hat{x}_0(t) = a_0\phi_0(t).$$

Find the value of a_0 that minimizes the energy in the error signal over the interval.

(h) We now wish to approximate e^t by a Taylor series using two terms—i.e., $\hat{x}_1(t) = a_0 + a_1 t$. Find the optimum values for a_0 and a_1. [*Hint:* Compute E in terms of a_0 and a_1, and then solve the simultaneous equations

$$\frac{\partial E}{\partial a_0} = 0 \quad \text{and} \quad \frac{\partial E}{\partial a_1} = 0.$$

Note that your answer for a_0 has changed from its value in part (g), where there was only one term in the series. Further, as you increase the number of terms in the series, that coefficient and all others will continue to change. We can thus see the advantage to be gained in expanding a function using orthogonal terms.]

3.67 As we discussed in the text, the origins of Fourier analysis can be found in problems of mathematical physics. In particular, the work of Fourier was motivated by his investigation of heat diffusion. In this problem, we illustrate how the Fourier series enter into the investigation.[13]

Consider the problem of determining the temperature at a given depth beneath the surface of the earth as a function of time, where we assume that the temperature at the surface is a given function of time $T(t)$ that is periodic with period 1. (The unit of time is one year.) Let $T(x, t)$ denote the temperature at a depth x below the surface at time t. This function obeys the heat diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{1}{2}k^2 \frac{\partial^2 T(x,t)}{\partial x^2} \qquad \text{(P3.67–1)}$$

with auxiliary condition

$$T(0, t) = T(t). \qquad \text{(P3.67–2)}$$

Here, k is the heat diffusion constant for the earth ($k > 0$). Suppose that we expand $T(t)$ in a Fourier series:

$$T(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn2\pi t}. \qquad \text{(P3.67–3)}$$

Similarly, let us expand $T(x, t)$ at any given depth x in a Fourier series in t. We obtain

$$T(x, t) = \sum_{n=-\infty}^{+\infty} b_n(x) e^{jn2\pi t}, \qquad \text{(P3.67– 4)}$$

where the Fourier coefficients $b_n(x)$ depend upon the depth x.

[13] The problem has been adapted from A. Sommerfeld, *Partial Differential Equations in Physics* (N. York: Academic Press, 1949), pp 68–71.

(a) Use eqs. (P3.67–1)–(P3.67–4) to show that $b_n(x)$ satisfies the differential equation

$$\frac{d^2 b_n(x)}{dx^2} = \frac{4\pi jn}{k^2} b_n(x) \qquad \text{(P3.67–5a)}$$

with auxiliary condition

$$b_n(0) = a_n. \qquad \text{(P3.67–5b)}$$

Since eq. (P3.67–5a) is a second-order equation, we need a second auxiliary condition. We argue on physical grounds that, far below the earth's surface, the variations in temperature due to surface fluctuations should disappear. That is,

$$\lim_{x \to \infty} T(x, t) = \text{a constant.} \qquad \text{(P3.67–5c)}$$

(b) Show that the solution of eqs. (P3.67–5) is

$$b_n(x) = \begin{cases} a_n \exp[-\sqrt{2\pi|n|}(1 + j)x/k], & n \geq 0 \\ a_n \exp[-\sqrt{2\pi|n|}(1 - j)x/k], & n \leq 0 \end{cases}.$$

(c) Thus, the temperature oscillations at depth x are damped and phase-shifted versions of the temperature oscillations at the surface. To see this more clearly, let

$$T(t) = a_0 + a_1 \sin 2\pi t$$

(so that a_0 represents the mean yearly temperature). Sketch $T(t)$ and $T(x, t)$ over a one-year period for

$$x = k\sqrt{\frac{\pi}{2}},$$

$a_0 = 2$, and $a_1 = 1$. Note that at this depth not only are the temperature oscillations significantly damped, but the phase shift is such that it is warmest in winter and coldest in summer. This is exactly the reason why vegetable cellars are constructed!

3.68. Consider the closed contour shown in Figure P3.68. As illustrated, we can view this curve as being traced out by the tip of a rotating vector of varying length. Let $r(\theta)$ denote the length of the vector as a function of the angle θ. Then $r(\theta)$ is periodic in θ with period 2π and thus has a Fourier series representation. Let $\{a_k\}$ denote the Fourier coefficients of $r(\theta)$.

(a) Consider now the projection $x(\theta)$ of the vector $r(\theta)$ onto the x-axis, as indicated in the figure. Determine the Fourier coefficients for $x(\theta)$ in terms of the a_k's.

(b) Consider the sequence of coefficients

$$b_k = a_k e^{jk\pi/4}.$$

Sketch the figure in the plane that corresponds to this set of coefficients.

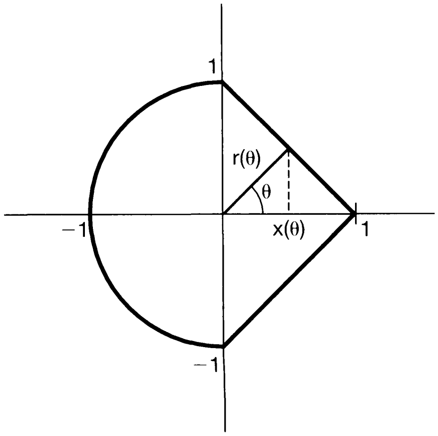


Figure P3.68

(c) Repeat part (b) with

$$b_k = a_k \delta[k].$$

(d) Sketch figures in the plane such that $r(\theta)$ is *not* constant, but does have each of the following properties:
(i) $r(\theta)$ is even.
(ii) The fundamental period of $r(\theta)$ is π.
(iii) The fundamental period of $r(\theta)$ is $\pi/2$.

3.69. In this problem, we consider the discrete-time counterpart of the concepts introduced in Problems 3.65 and 3.66. In analogy with the continuous-time case, two discrete-time signals $\phi_k[n]$ and $\phi_m[n]$ are said to be *orthogonal* over the interval (N_1, N_2) if

$$\sum_{n=N_1}^{N_2} \phi_k[n]\phi_m^*[n] = \begin{cases} A_k, & k = m \\ 0, & k \neq m \end{cases}. \qquad \text{(P3.69–1)}$$

If the value of the constants A_k and A_m are both 1, then the signals are said to be *orthonormal*.

(a) Consider the signals

$$\phi_k[n] = \delta[n-k],\ k = 0, \pm 1, \pm 2, \ldots, \pm N.$$

Show that these signals are orthonormal over the interval $(-N, N)$.

(b) Show that the signals

$$\phi_k[n] = e^{jk(2\pi/N)n},\ k = 0, 1, \ldots, N-1,$$

are orthogonal over any interval of length N.

(c) Show that if

$$x[n] = \sum_{i=1}^{M} a_i \phi_i[n],$$

where the $\phi_i[n]$ are orthogonal over the interval (N_1, N_2), then

$$\sum_{n=N_1}^{N_2} |x[n]|^2 = \sum_{i=1}^{M} |a_i|^2 A_i.$$

(d) Let $\phi_i[n], i = 0, 1, \ldots, M$, be a set of orthogonal functions over the interval (N_1, N_2), and let $x[n]$ be a given signal. Suppose that we wish to approximate $x[n]$ as a linear combination of the $\phi_i[n]$; that is,

$$\hat{x}[n] = \sum_{i=0}^{M} a_i \phi_i[n],$$

where the a_i are constant coefficients. Let

$$e[n] = x[n] - \hat{x}[n],$$

and show that if we wish to minimize

$$E = \sum_{n=N_1}^{N_2} |e[n]|^2,$$

then the a_i are given by

$$a_i = \frac{1}{A_i} \sum_{n=N_1}^{N_2} x[n]\phi_i^*[n]. \tag{P3.69–2}$$

[*Hint:* As in Problem 3.66, express E in terms of a_i, $\phi_i[n]$, A_i, and $x[n]$, write $a_i = b_i + jc_i$, and show that the equations

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0$$

are satisfied by the a_i given by eq. (P3.69–2). Note that applying this result when the $\phi_i[n]$ are as in part (b) yields eq. (3.95) for a_k.]

(e) Apply the result of part (d) when the $\phi_i[n]$ are as in part (a) to determine the coefficients a_i in terms of $x[n]$.

3.70. (a) In this problem, we consider the definition of the two-dimensional Fourier series for periodic signals with two independent variables. Specifically, consider a signal $x(t_1, t_2)$ that satisfies the equation

$$x(t_1, t_2) = x(t_1 + T_1, t_2 + T_2), \text{ for all } t_1, t_2.$$

This signal is periodic with period T_1 in the t_1 direction and with period T_2 in the t_2 direction. Such a signal has a series representation of the form

$$x(t_1, t_2) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} a_{mn} e^{j(m\omega_1 t_1 + n\omega_2 t_2)},$$

where

$$\omega_1 = 2\pi/T_1, \quad \omega_2 = 2\pi/T_2.$$

Find an expression for a_{mn} in terms of $x(t_1, t_2)$.

(b) Determine the Fourier series coefficients a_{mn} for the following signals:
(i) $\cos(2\pi t_1 + 2t_2)$
(ii) the signal illustrated in Figure P3.70

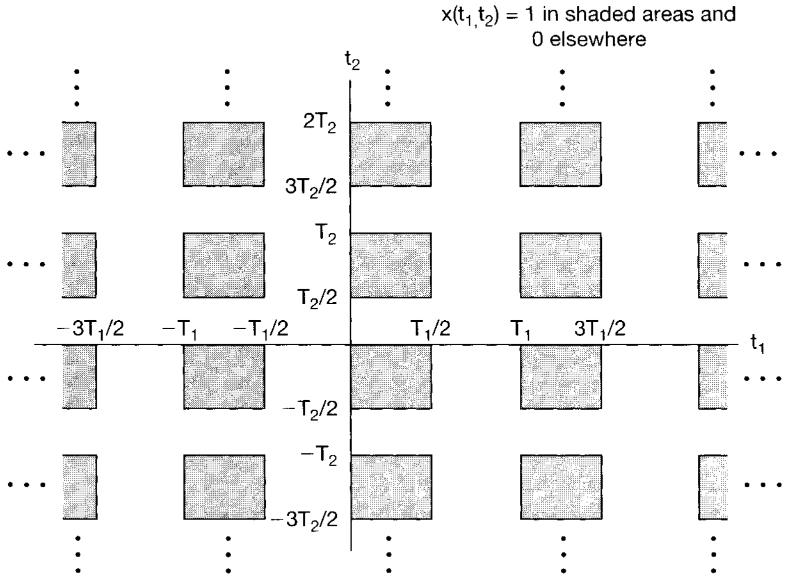


Figure P3.70

3.71. Consider the mechanical system shown in Figure P3.71. The differential equation relating velocity $v(t)$ and the input force $f(t)$ is given by

$$Bv(t) + K\int v(t)\,dt = f(t).$$

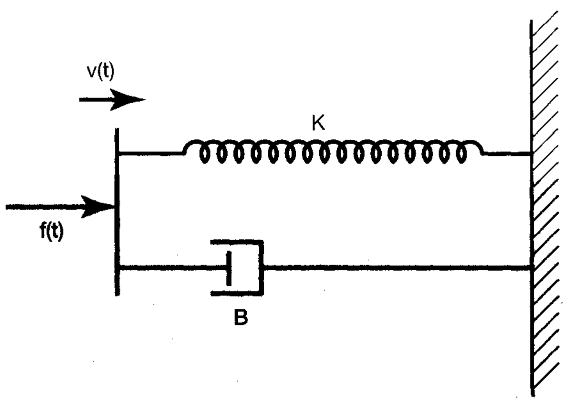


Figure P3.71

(a) Assuming that the output is $f_s(t)$, the compressive force acting on the spring, write the differential equation relating $f_s(t)$ and $f(t)$. Obtain the frequency response of the system, and argue that it approximates that of a lowpass filter.

(b) Assuming that the output is $f_d(t)$, the compressive force acting on the dashpot, write the differential equation relating $f_d(t)$ and $f(t)$. Obtain the frequency response of the system, and argue that it approximates that of a highpass filter.