Midterm Examination 2
ECE 301
Division 3, Fall 2007
Instructor: Mimi Boutin

Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.

2. You have 50 minutes to complete the 5 questions contained in this exam. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.

3. This booklet contains 10 pages. The last four pages contain a table of formulas and properties. You may tear out these pages **once the exam begins**. TABLE USE RULES: You may use any fact contained in the table without justification. Simply write the number of the corresponding table item to indicate which fact you are using from the table. If you use a non-trivial fact that is *not* contained in the table, you must justify (i.e., prove) it in order to get full credit.

4. This is a closed book exam. Calculators, cell phones, and i-pods are strictly forbidden.

Name:____

Email:____

Signature:______________________________________

Itemized Scores

Problem 1:
Problem 2:
Problem 3:
Problem 4:
Problem 5:
Total:

(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{2j}\right)^{|n|}.$$

(20 pts) **2.** The Frequency response of a continuous-time LTI system is

$$H(j\omega) = \mathcal{H}(\omega) = \frac{1}{j\omega + 2}.$$

Use the convolution property of the Fourier transform to determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$.

(15 pts) **3.** True/False? The Fourier transform of a DT signal $x[n]$ is a periodic function, no matter what $x[n]$ is. (Justify your answer.)

(10 pts) **4.** A continuous-time LTI system has frequency response

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}.$$

Derive a differential equation representing this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

(10 pts) **5.** A CT signal $x(t)$ has Fourier transform

$$\mathcal{X}(\omega) = -2e^{(j-1)\omega}u(\omega+1).$$

Denote by $y(t)$ the signal obtained by delaying $x(t)$ by six seconds. Sketch a graph representing the magnitude $|\mathcal{Y}(\omega)|$ of the Fourier transform $\mathcal{Y}(\omega)$ of $y(t)$. (Justify your answer.)

Table

1 Definition of the Continuous-time Fourier Transform

Let $x(t)$ be a signal and denote by $\mathcal{X}(\omega)$ its Fourier transform.

$$\begin{aligned} \text{Fourier Transform: } \mathcal{X}(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt && (1) \\ \text{Inverse Fourier Transform: } x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{X}(\omega)e^{j\omega t}d\omega && (2) \end{aligned}$$

2 Some Continuous-time Fourier Transforms

$$\begin{aligned} e^{j\omega_0 t} &\xrightarrow{\mathcal{F}} 2\pi\delta(\omega-\omega_0) && (3) \\ 1 &\xrightarrow{\mathcal{F}} 2\pi\delta(\omega) && (4) \\ \frac{\sin Wt}{\pi t} &\xrightarrow{\mathcal{F}} u(\omega+W)-u(\omega-W) && (5) \\ \delta(t) &\xrightarrow{\mathcal{F}} 1 && (6) \\ e^{-at}u(t), \mathcal{R}e\{a\}>0 &\xrightarrow{\mathcal{F}} \frac{1}{a+j\omega} && (7) \\ \sum_{n=-\infty}^{\infty}\delta(t-nT) &\xrightarrow{\mathcal{F}} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi k}{T}) && (8) \end{aligned}$$

3 Properties of the Continuous-time Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

$$\begin{aligned}
\text{Linearity: } ax(t)+by(t) &\xrightarrow{\mathcal{F}} a\mathcal{X}(\omega)+b\mathcal{Y}(\omega) && (9)\\
\text{Time Shifting: } x(t-t_0) &\xrightarrow{\mathcal{F}} e^{-j\omega t_0}\mathcal{X}(\omega) && (10)\\
\text{Frequency Shifting: } e^{j\omega_0 t}x(t) &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega-\omega_0) && (11)\\
\text{Conjugation: } x^*(t) &\xrightarrow{\mathcal{F}} \mathcal{X}^*(-\omega) && (12)\\
\text{Scaling: } x(at) &\xrightarrow{\mathcal{F}} \frac{1}{|a|}\mathcal{X}\left(\frac{\omega}{a}\right) && (13)\\
\text{Multiplication: } x(t)y(t) &\xrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathcal{X}(\omega)*\mathcal{Y}(\omega) && (14)\\
\text{Convolution: } x(t)*y(t) &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\mathcal{Y}(\omega) && (15)\\
\text{Differentiation in Time: } \frac{d}{dt}x(t) &\xrightarrow{\mathcal{F}} j\omega\mathcal{X}(\omega) && (16)\\
x(t)\text{real} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)=\mathcal{X}^*(-\omega) && (17)\\
x(t)\text{real and even} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{real and even} && (18)\\
x(t)\text{real and odd} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{pure imaginary and odd} && (19)\\
\text{Parseval's Relation: } \int_{-\infty}^{\infty}|x(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{X}(\omega)|^2\,d\omega && (20)
\end{aligned}$$

4 Fourier Series of Continuous-time Periodic Signals with period T

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} && (21)\\
a_k &= \frac{1}{T}\int_0^T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt && (22)
\end{aligned}$$

5 Fourier Series of Discrete-time Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \tag{23}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk(\frac{2\pi}{N})n} \tag{24}$$

6 Definition of the Discrete-time Fourier Transform

Let $x[n]$ be a discrete-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform.

$$\text{Fourier Transform: } \mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{25}$$

$$\text{Inverse Fourier Transform: } x[n] = \frac{1}{2\pi}\int_{2\pi} \mathcal{X}(\omega) e^{j\omega n} d\omega \tag{26}$$

7 Some Discrete-time Fourier Transforms

$$\sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \tag{27}$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \tag{28}$$

$$\frac{\sin Wn}{\pi n}, 0 < W < \pi \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) = \begin{cases} 1, & 0 \leq |\omega| < W \\ 0, & \pi \geq |\omega| > W \end{cases} \tag{29}$$

$$\mathcal{X}(\omega) \text{periodic with period } 2\pi$$

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \tag{30}$$

$$\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{1 - \alpha e^{-j\omega}} \tag{31}$$

$$(n+1)\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{(1 - \alpha e^{-j\omega})^2} \tag{32}$$

8 Properties of the Discrete-time Fourier Transform

Let $x[n]$ and $y[n]$ be DT signals. Denote by $\mathcal{X}(\omega)$ and $\mathcal{Y}(\omega)$ their Fourier transforms.

$$\begin{aligned}
\text{Linearity: } ax[n]+by[n] &\xrightarrow{\mathcal{F}} a\mathcal{X}(\omega)+b\mathcal{Y}(\omega) && (33)\\
\text{Time Shifting: } x[n-n_0] &\xrightarrow{\mathcal{F}} e^{-j\omega n_0}\mathcal{X}(\omega) && (34)\\
\text{Frequency Shifting: } e^{j\omega_0 n}x[n] &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega-\omega_0) && (35)\\
\text{Conjugation: } x^*[n] &\xrightarrow{\mathcal{F}} \mathcal{X}^*(-\omega) && (36)\\
\text{Time Reversal: } x[-n] &\xrightarrow{\mathcal{F}} \mathcal{X}(-\omega) && (37)\\
x_k[n]=\begin{cases} x[\frac{n}{k}], & \text{if } k \text{ divides } n\\ 0, & \text{else.}\end{cases} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega) && (38)\\
\text{Multiplication: } x[n]y[n] &\xrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathcal{X}(\omega)*\mathcal{Y}(\omega) && (39)\\
\text{Convolution: } x(t)*y(t) &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\mathcal{Y}(\omega) && (40)\\
\text{Differentiation: } x[n]-x[n-1] &\xrightarrow{\mathcal{F}} (1-e^{-j\omega})\mathcal{X}(\omega) && (41)\\
\text{Accumulation: } \sum_{k=-\infty}^{n} x[k] &\xrightarrow{\mathcal{F}} \frac{\mathcal{X}(\omega)}{1-e^{-j\omega}}\\
&\qquad +\pi\mathcal{X}(0)\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k) && (42)\\
x[n]\text{real} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)=\mathcal{X}^*(-\omega) && (43)\\
x[n]\text{real and even} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{real and even} && (44)\\
x[n]\text{real and odd} &\xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{pure imaginary and odd} && (45)\\
\text{Parseval's Relation: } \sum_{n=-\infty}^{\infty}|x[n]|^2 &= \frac{1}{2\pi}\int_{2\pi}|\mathcal{X}(w)|^2\,d\omega && (46)
\end{aligned}$$