Assignment 3 - Integration & Residues

Exercises adorned with a $\star$ were taken from past quals written by Prof. Bell.

1. $\star$ Suppose a_n is a sequence of distinct non-zero complex numbers satisfying
$$\sum_{n=1}^{\infty} |a_n|^{-1} < \infty.$$
Let $\mathcal{A} = \{a_n \mid n = 1, \cdots, \infty\}$.
   (a) Prove that $\sum_{n=1}^{\infty} \frac{1}{z-a_n}$ converges to a function $f(z)$ that is analytic on $\mathbb{C} - \mathcal{A}$.
   (b) For $z \in \mathbb{C} - \mathcal{A}$, let
$$G(z) = \exp\left(\int_{\gamma_0^z} f(w)\, dw\right)$$
where γ_0^z is a curve in $\mathbb{C} - \mathcal{A}$ that starts at the origin and ends at z. Prove that G is well-defined and analytic on $\mathbb{C} - \mathcal{A}$. Show that G has removable singularities at each point a_n. Finally, show that the points a_n are in fact simple zeroes of G.

2. Compute
$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^3}\, dx.$$

3. Evaluate
$$\int_{0}^{\infty} \frac{\log x}{(x^2 + 1)^2}\, dx.$$

4. $\star$ Suppose P and Q are polynomials with the degree of P at least two less than the degree of Q. Prove that the sum of the residues of P/Q in the complex plane is zero.

5. Evaluate
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)(x^2 + 4)}\, dx$$
(and show your work).

6. Show that
$$\int_{-\infty}^{\infty} \frac{x \sin(2x)}{x^4 + 16}\, dx = \frac{\pi e^{-2\sqrt{2}} \sin(2\sqrt{2})}{4}.$$
Show all work.

7. Evaluate
$$\int_{-\infty}^{\infty} \frac{\cos t}{1+t^4}\, dt.$$

8. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\sin(\operatorname{Log} x)}{x^2+4}\, dx$$
where Log denote the principal value of the logarithm (real on the positive ray).

9. $\star$ Compute
$$\int_{0}^{\infty} \frac{1}{x^3+1}\, dx.$$