10/8 Example

Show that the FT of $\cos 2\pi t$ is

is $\pi\,\delta(\omega+2\pi) + \pi\,\delta(\omega-2\pi)$

We show that

I.F.T. of $\pi\,\delta(\omega+2\pi) + \pi\,\delta(\omega-2\pi)$ is $\cos 2\pi t$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\,\delta(\omega+2\pi) + \pi\,\delta(\omega-2\pi)\right)e^{j\omega t}\,d\omega =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega+2\pi)\,e^{j\omega t}\,d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega-2\pi)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{2}e^{-j2\pi t} + \frac{1}{2}e^{j2\pi t} = \cos(2\pi t)$$

Example

Easy question

The input of an LTI system is $x(t) = 1$
The unit impulse response of the system is

$$h(t) = \delta(t-3)$$

What is the F.T. of the response $y(t)$ of the system?

$$Y(\omega) = \mathcal{F}(x(t) * h(t))$$

$$= \mathcal{F}(x(t))\,\mathcal{F}(h(t))$$

$$= \mathcal{F}(1)\,\mathcal{F}(\delta(t-3))$$

$\downarrow$ you can do this since $\mathcal{F}^{-1}(\delta(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega) e^{j\omega t}\,dt$

$= e^0 = 1$

$\therefore \mathcal{F}(1) = 2\pi\,\delta(\omega)$

$$= 2\pi\,\delta(\omega)\, e^{-3j\omega}\,\mathcal{F}(\delta(t))$$

$$= \{2\pi\,\delta(\omega)\} e^{-3j\omega} \cdot 1$$

$$= 2\pi\,\delta(\omega)\, e^{0}$$

$$= 2\pi\,\delta(\omega)$$ final solution

* Exercise: compute the F.T. of $u(t-3)$

Frequency Response

$H(j\omega)$ in CT
$H(e^{j\omega})$ in DT

$$e^{st} \rightarrow \boxed{} \rightarrow H(s)\, e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

when $s = j\omega$
$H(j\omega)$

$$e^{jk\omega_0 t} \rightarrow \boxed{} \rightarrow \underline{H(jk\omega_0)}\, e^{jk\omega_0 t}$$

this shows us what the system does to frequency components of the signal

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$H(j\omega) = \underbrace{\int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt}_{\text{F.T. of } h(t)}$$

$\mathcal{H}(\omega)$

$$X(\omega) \rightarrow \boxed{h(t)} \rightarrow X(\omega) \cdot \underbrace{\mathcal{H}(\omega)}_{= H(j\omega)}$$

$$X(\omega) \rightarrow \boxed{h(t)} \rightarrow X(\omega)H(j\omega)$$

Example Question

What is the F.T. of the LTI system defined by

$$y(t) = \frac{d}{dt}x(t)$$

$$\mathcal{F}(y(t)) = \mathcal{F}\left(\frac{d}{dt}x(t)\right)$$

$Y(\omega) = j\omega X(\omega)$ by the diff. property of FTs

we know that:

note
s is specified as $s = j\omega$

$$Y(\omega) = H(j\omega)X(\omega)$$

So $H(j\omega) = j\omega$ is the f response to the system

Q2 what is the F.T. of $h(t)$?

$$\mathcal{F}(h(t)) = j\omega$$

$$H(\omega) = j\omega \quad \boxed{\cancel{H(j\omega) = j\,j\omega = -\omega}}$$

this is not the F.T.

Systems characterized by linear, constant coefficient differential equations

what is this?

$$\hookrightarrow \quad \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t)$$

Example Question

What is the f response of this system?

FT on left = FT on right

$$\mathcal{F}\left(\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right) = \mathcal{F}\left(\sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t)\right)$$

$$\Leftrightarrow \quad \sum_{k=0}^{N} a_k \mathcal{F}\left(\frac{d^k}{dt^k} y(t)\right) = \sum_{k=0}^{m} b_k \mathcal{F}\left(\frac{d^k}{dt^k} x(t)\right)$$

apply diff property k times

$$\Leftrightarrow \quad \sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{m} b_k (j\omega)^k X(\omega)$$

$$\Leftrightarrow \quad Y(\omega) = \underbrace{\frac{\sum_{k=0}^{m} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}}_{H(j\omega) \equiv f \text{ response}} X(\omega)$$

this is true because

$$Y(\omega) = H(j\omega) X(\omega)$$

cont'd

b) find $h(t)$

since $H(j\omega) = \mathcal{H}(\omega) = \mathcal{F}(h(t))$

$$\Rightarrow h(t) = \mathcal{F}^{-1}(H(j\omega))$$

* exercise:

$$\frac{d\,y(t)}{dt} + 4y(t) = x(t)$$

a) what is the f response of system?
b) what is the unit impulse response ($h(t)$) of the system

<u>multiplication property</u>

$$\mathcal{F}(x_1(t)\cdot x_2(t)) = \frac{1}{2\pi}\, X_1(\omega) * X_2(\omega)$$

this is the dual of the convolution property

Note on HW6:

Do all book problems and
hand in <u>half</u> by Wed 15^{th} by 5:30
or do all on Rhea

Chapter 5 DT Fourier Transforms

$$x[n] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(\omega)$$ this is continuous on w and periodic

if the signal is periodic then a Fourier Series will suffice

if the signal is aperiodic then the Fourier Transform must be applied but it may also be applied to periodic signals.

formulas

$$X(\omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n}$$