Ex Two types of chips can be used to ~~mad~~ manufacture a certain system.
The lifetime X in hours of the system given chip i $(i=1,2)$ has the pdf
Chip 1 is twice as likely to be used

$$f_X(x \mid C_i) = \lambda_i e^{-\lambda_i x} u(x), \quad \lambda_i > 0, \quad i=1,2$$

a) Find the pdf of X

$$f_X(x) = f_X(x \mid C_1)\Pr(C_1) + f_X(x \mid C_2)\Pr(C_2)$$

$$= \left(\tfrac{2}{3}\lambda_1 e^{-\lambda_1 x} + \tfrac{1}{3}\lambda_2 e^{-\lambda_2 x}\right) u(x)$$

b) Find the probability that ~~the chip will last at least~~ chip 2 is used given that the system has lasted at least t hours.

$$\Pr(C_2 \mid X > t) = \frac{\Pr(X > t \mid C_2)\Pr(C_2)}{\Pr(X > t)}$$

$$= \frac{\tfrac{1}{3}\int_t^\infty f_X(x \mid C_2)\,dx}{\int_t^\infty f_X(x)\,dx}$$

$$= \frac{\tfrac{1}{3}\cdot e^{-\lambda_2 t}}{\tfrac{2}{3}e^{-\lambda_1 t} + \tfrac{1}{3}e^{-\lambda_2 t}}$$

c) Find the mean lifetime of the system.

$$E[X] = E[X|C_1]\Pr(C_1) + E[X|C_2]\Pr(C_2)$$

$$E[X|C_i] = \int_0^\infty x\lambda_i e^{-\lambda_i x}\,dx = \frac{1}{\lambda_i}$$

$$\uparrow \int_{-\infty}^{\infty} x f_x(x|C_i)\,dx$$

$$E[X] = 2/3 \cdot \frac{1}{\lambda_1} + 1/3 \cdot \frac{1}{\lambda_2}$$

Note: What is the total variance formula? Define

$$Var[X|M_j] = E[(X - E[X|M_j])^2 \mid M_j]$$

$$Var[X] \neq \sum_{j=1}^{n} Var[X|M_j]\Pr(M_j)$$

With total expectation ~~a~~ and $g(X) = (X - E[X])^2$

$$Var[X] = \sum_{j=1}^{n} E[(X - E[X])^2 | M_j]\Pr(M_j)$$

$$= \sum_{j=1}^{n} \left(Var[X|M_j] + (E[X|M_j] - E[X])^2\right)\Pr(M_j)$$

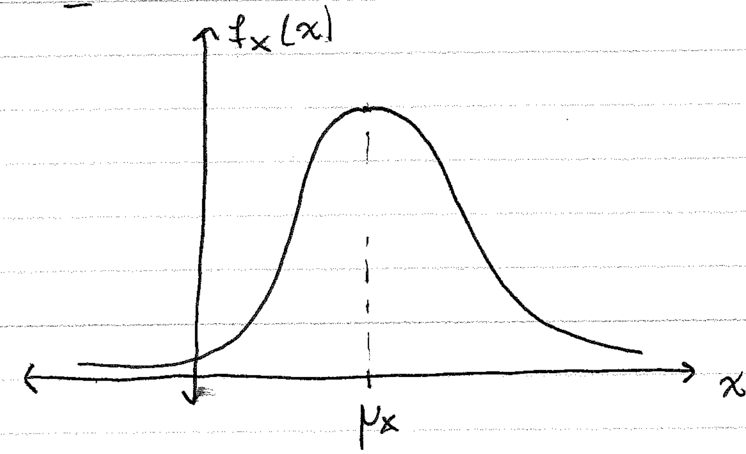
Gaussian Random Variable

A Gaussian (normal) ~~rand~~ r.v. X has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi \sigma_X^2}} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right), \quad -\infty < x < \infty$$

Turns out that μ_X and σ_X^2 are really the mean and variance of X

Pf: Left as an exercise



Gaussian r.v.s are used to model physical quantities that are the sum of a large number of independent processes (noise, measurement error). Also model results of statistical samples and surveys.

The Gaussian (or normal) cdf is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi \sigma_X^2}} \exp\left(-\frac{(x'-\mu_X)^2}{2\sigma_X^2}\right) dx'$$

$$z = \frac{x'-\mu_X}{\sigma_X}, \quad dz = \frac{dx'}{\sigma_X}$$

$$F_X(x) = \int_{-\infty}^{\frac{x-\mu_X}{\sigma_X}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \Phi\left(\frac{x-\mu_X}{\sigma_X}\right)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

<u>Note</u>: $\Phi(x)$ is the cdf of Gaussian r.v. with $\mu_X = 0$ and $\sigma_X^2 = 1$

$\Phi(x)$ is tabulated for $x \geq 0$

$\Phi(x)$ can be computed for $x < 0$ from

$$\Phi(x) = 1 - \Phi(-x)$$

Sometimes $Q(x)$ is used instead of $\Phi(x)$

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$= 1 - \Phi(x)$$

$$= 1 - Q(-x)$$

<u>Note</u>: $Q(x)$ is $Pr(X > x)$ for a Gaussian r.v. with $\mu_X = 0$ and $\sigma_X^2 = 1$

Ex: Let X be a Gaussian r.v. with
$\mu_X = 3$, $\sigma_X^2 = 9$.

a) Find $Pr(X>0)$

$$Pr(X>0) = 1 - F_X(0)$$
$$= 1 - \Phi\left(\frac{0-3}{3}\right)$$
$$= 1 - \Phi(-1) = \Phi(1) \approx 0.8413$$

b) $$Pr(2<X<5) = F_X(5) - F_X(2)$$
$$= \Phi\left(\frac{5-3}{3}\right) - \Phi\left(\frac{2-3}{3}\right)$$
$$= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \approx 0.3799$$

c) $$Pr(|X-3|>6) = Pr(X<-3) + Pr(X>9)$$
$$= F_X(-3) + (1 - F_X(9))$$
$$= \Phi(-2) + (1 - \Phi(2))$$
$$= 2(1 - \Phi(2)) \approx 0.0456$$

Note: $Y = aX + b$, X is Gaussian with mean μ_X and variance σ_X^2

$\Rightarrow$ Y is Gaussian

$$\mu_Y = E[aX+b] = a\mu_X + b$$
$$\sigma_Y^2 = Var[aX+b] = a^2\sigma_X^2$$

Higher Order Central Moments of Gaussians R.V

Gaussian r.v.s are completely specified by their mean and variance. Let X be a Gaussian r.v. with mean μ_X and variance σ_X^2. All higher order moments of X can be found in terms of μ_X and σ_X^2.

The higher order central moments of X are given by

$$E[(X-\mu_X)^n] = \begin{cases} 0 & , \ n \text{ is odd} \\ 1\cdot 3\cdot\ldots\cdot(n-1)\,\sigma_X^n & , \ n \text{ is even} \end{cases}$$

The higher moments of X can be found by using the expression above as follows.

Ex The fourth central moment of X is given by

$$E[(X-\mu_X)^4] = 3\sigma_X^4$$

The fourth moment of X is given by

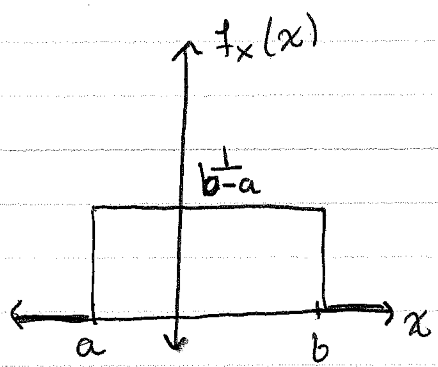
$$\begin{aligned} E[X^4] &= E[X^4 - (X-\mu_X)^4 + (X-\mu_X)^4] \\ &= 3\sigma_X^4 + E[X^4 - (X-\mu_X)^4] \\ &= 3\sigma_X^4 + E[4X^3\mu_X - 6X^2\mu_X^2 + 4X\mu_X^3 - \mu_X^4] \\ &\ \ \vdots \\ &= 3\sigma_X^4 + 6\sigma_X^2\mu_X^2 + \mu_X^4 \end{aligned}$$

Other Important Continuous R.V.s

- **Uniform R.V.**

A r.v. X uniform on $[a,b]$ has pdf

$$f_X(x) = \begin{cases} \cancel{b-a}\ \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{else} \end{cases}$$



$$E[X] = \frac{a+b}{2}$$

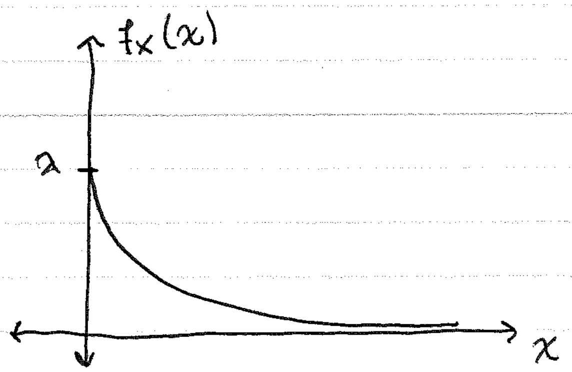
$$Var[X] = \frac{(b-a)^2}{2}$$

Uniform r.v.s are used to model quantities that are equally likely to take on any value in a given interval of the real line.

- **Exponential R.V.**

An exponential r.v. X with parameter λ (mean / average $1/\lambda$) has pdf

$$f_X(x) = \lambda e^{-\lambda x} u(x), \quad \lambda > 0$$



$$E[X] = \frac{1}{\lambda}$$

$$Var[X] = \frac{1}{\lambda^2}$$