Case 2 : X, Y are jointly continuous

Will only consider cases where $(g(x,y), h(x,y))$ are smooth and 1:1. In this case, the equations $u = g(x,y)$ and $v = h(x,y)$ can be uniquely solved for x and y in terms of u and v, and the Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} \neq 0 \quad \text{for any } x \text{ and } y.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

~~exist~~ exists every where when (g,h) are smooth.

~~Defn.~~ Density Method

Let X, Y be r.v.s and $U = g(X,Y)$, $V = h(X,Y)$ where (g,h) are smooth and 1:1. Then

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1}, \quad (u,v) = (g(x,y), h(x,y))$$

$$= 0 \quad , \text{ else}$$

Ex. Let X, Y be r.v.s and

$$U = X + Y, \quad V = X - Y.$$

Want to find $f_{U,V}(u,v)$

Density Method:

From the equation $u = x + y$ and $v = x - y$

we have that $x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left\| \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\| = -2$$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1}$$

$$= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2} \right) \cdot \frac{1}{2}$$

Auxiliary Variable Method

Suppose just have $U = g(X,Y)$ and want $f_U(u)$.

Can choose $V = h(X,Y)$ and use the density method to find $f_{U,V}(u,v)$, then find $f_U(u)$ as

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv$$

Choose $V = h(X, Y)$ to simplify either (or both):

1. $\left| \dfrac{\partial(u, v)}{\partial(x, y)} \right|$

2. $\int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv$

<u>Ex</u> X, Y jointly continuous r.v.s.

Let $U = X + Y$. Find $f_U(u)$.

<u>Density Method</u>

Let $V = h(X, Y) = Y$

Then $u = x + y$, $v = y$ $\Rightarrow$ $x = u - v$, $y = v$

$$\frac{\partial(u, v)}{\partial(x, y)} = \left| \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right| = 1$$

$$f_{U,V}(u, v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left| \frac{\partial(u, v)}{\partial(x, y)} \right|^{-1}$$

$$= f_{X,Y}(u - v, v)$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}(u - v, v)\, dv$$

$\underline{\text{EX}}$ Let X, Y be r.v.s with jpdf

$$f_{X,Y}(x, y) = \begin{cases} 6x & , \quad 0 \le x + y \le 1 \ , \ x \ge 0 \ , \ y \ge 0 \\ 0 & , \quad \text{else} \end{cases}$$

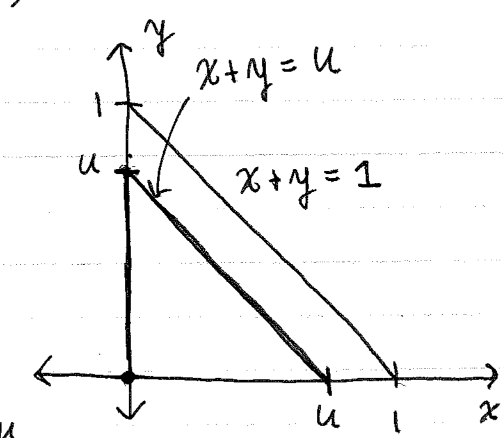
Let $U = X + Y$. Find $f_U(u)$.

Density Method

Let $V = Y$.

$u = x + y \ , \quad v = y \quad \Rightarrow \quad x = u - v \ , \quad y = v$

$f_{U,V}(u, v) = f_{X,Y}(u - v, v)$



$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(u - v, v)\, dv$$

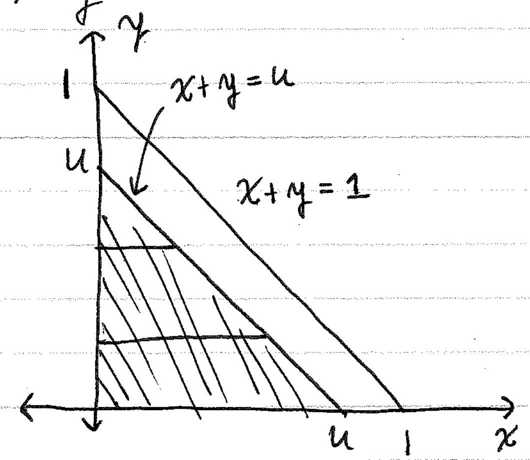
$$= \int_0^u 6(u - v)\, dv \ , \quad 0 \le u \le 1$$

$$= 3u^2 \ , \quad 0 \le u \le 1$$

$$= 0 \ , \quad \text{else}$$

Distribution Method

$$F_U(u) = \iint_{x+y \le u} f_{X,Y}(x,y)\,dx\,dy$$



$$= \int_0^u \int_0^{u-y} 6x\,dx\,dy$$

$$= \cancel{\int_0^u}\ u^3 \quad , \ 0 \le u \le 1$$

$$F_U(u) = \begin{cases} 0 & , \quad u \le 0 \\ u^3 & , \quad 0 \le u \le 1 \\ 1 & , \quad u \ge 1 \end{cases}$$

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \begin{cases} 3u^2 & , \quad 0 \le u \le 1 \\ 0 & , \quad \text{else} \end{cases}$$