Probability Density Function

The probability density function (pdf) of a r.v. X is

$$f_X(x) = \frac{d}{dx} F_X(x), \quad -\infty < x < \infty$$

or equivalently

$$F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx', \quad -\infty < x < \infty$$

Note: When $F_X(x)$ has jumps, $f_X(x)$ will have impulses.

The pdf $f_X(x)$ represents the "density" of probability on a small interval around x.

$$f_X(x) = \lim_{h \to 0} \frac{F_X(x+h) - F_X(x)}{h}$$

$$= \lim_{h \to 0} \frac{\Pr(x < X \le x+h)}{h}$$

$$\Rightarrow \quad f_X(x) \cdot h \approx \Pr(x < X \le x+h) \quad \text{for small } h$$

Properties of pdf

1) $f_X(x) \geq 0$

2) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

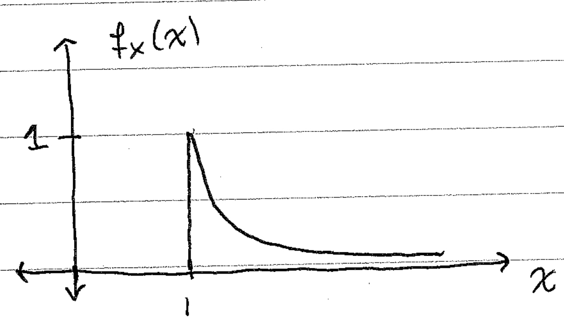
3) $Pr(X \in A) = \int_A f_X(x)\,dx$, where $A \subset \mathbb{R}$

Ex Let X be a r.v. with pdf

$$f_X(x) = \begin{cases} 1/x^2 , & x \geq 1 \\ 0 , & x < 1 \end{cases}$$

$$f_X(x) = \frac{1}{x^2} u(x-1)$$

a) Find $Pr(X \leq 3)$, $Pr(X > 2)$, $Pr(1 < X \leq 4)$
$Pr(X < 2 \text{ or } X \geq 4)$



X is continuous r.v.

$$Pr(X \in A) = \int_A f_X(x)\,dx$$

$$
\begin{aligned}
\Pr(X \le 3) &= \Pr(X \in (-\infty, 3]) \\
&= \int_{-\infty}^{3} f_X(x)\,dx \\
&= \int_{1}^{3} \frac{1}{x^2}\,dx \\
&= \left(-\frac{1}{x}\right)\Big|_1^3 \\
&= 1 - 1/3 = 2/3
\end{aligned}
$$

$$
\begin{aligned}
\Pr(X > 2) &= \Pr(X \in (2, \infty)) \\
&= \int_{2}^{\infty} \frac{1}{x^2}\,dx \\
&= \frac{1}{2} - 0 = 1/2
\end{aligned}
$$

$$
\begin{aligned}
\Pr(1 < X \le 4) &= \Pr(X \in (1, 4]) \qquad \cancel{\Pr(1 < X \le 4)} \\
&= \int_{1}^{4} f_X(x)\,dx \\
&= \left(-\frac{1}{x}\right)\Big|_1^4 = 1 - 1/4 = 3/4
\end{aligned}
$$

$$
\begin{aligned}
\Pr(X < 2 \text{ or } X \ge 4) &= \Pr(X < 2) + \Pr(X \ge 4) \\
&= 1 - \Pr(X \ge 2) + \int_4^{\infty} f_X(x)\,dx \\
&= 1 - \Pr(X > 2) + \left(-\frac{1}{x}\right)\Big|_4^{\infty} \\
&= 1 - 1/2 + 1/4 = 3/4
\end{aligned}
$$

Since X is continuous, $\Pr(X = 2) = 0$

b) Find the cdf of X

$$F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx'$$

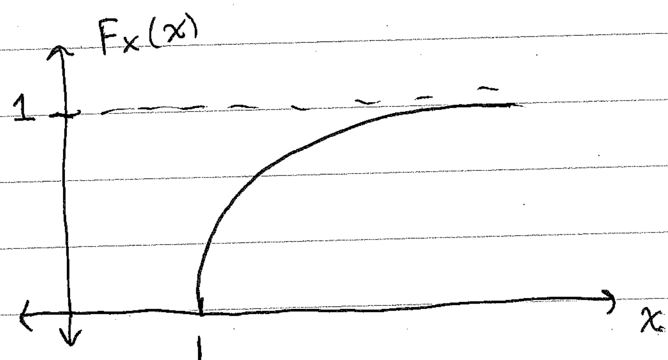
$$= \int_{1}^{x} \frac{1}{(x')^2}\,dx' \quad , \qquad x \geq 1$$

$$= \left(-\frac{1}{x}\right)\Big|_{1}^{x}$$

$$= 1 - \frac{1}{x}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx' = 0 \quad , \quad x < 1$$

$$F_X(x) = \begin{cases} 1 - 1/x & , \quad x \geq 1 \\ 0 & , \quad x < 1 \end{cases}$$



Note: The value of $f_X(x)$ can be assigned arbitrarily at a finite or countable number of points, provided it has no impulses at those points.

Does not change

$$\Pr(X \in A) = \int_A f_X(x)\,dx \quad \text{for any } A \subset \mathbb{R}$$

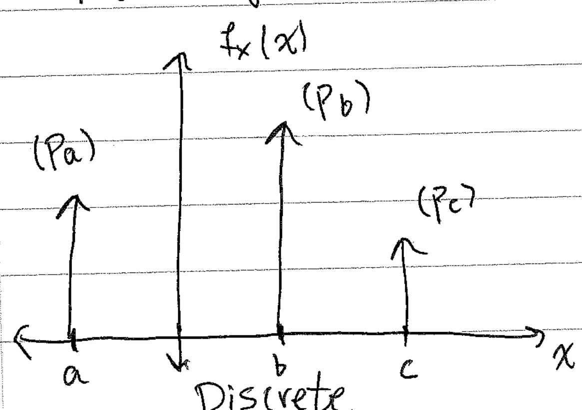
$$f_X(x) = \begin{cases} 1/x^2, & x > 1 \\ 100, & x = 1 \\ 0, & x < 1 \end{cases}$$

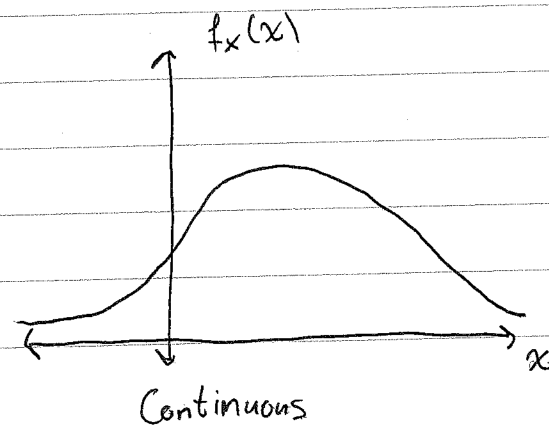
$$\Pr(X > 0) = \int_0^1 0\,dx + \underbrace{\cancel{\int_1^1 100\,dx}}_{0} + \int_1^\infty \frac{1}{x^2}\,dx$$

Again, we can describe discrete and continuous r.v.s in terms of pdf.

A discrete r.v. has a pdf only composed of impulses.

A continuous r.v. has a pdf that has no impulses.





Note: $$\Pr(X = x_0) = \lim_{\varepsilon \to 0^+} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f_X(x)\,dx$$

$$= \text{area of impulse at } x = x_0$$

For discrete r.v. X

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx} \sum_{x_i} p_X(x_i)\, u(x - x_i)$$

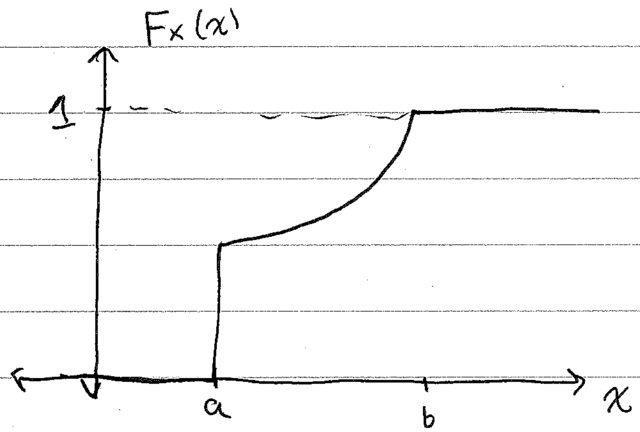
$$= \sum_{x_i} p_X(x_i) \frac{d}{dx} u(x - x_i)$$

$$= \sum_{x_i} p_X(x_i)\, \delta(x - x_i)$$

Mixed Random Variables

(finite)

Some cdfs may have jumps on a countable set of points $x_0, x_1, \ldots$ but may also increase continuously on at least one interval.



The cdf of the r.v.s can be expressed as

$$F_X(x) = p F_1(x) + (1-p) F_2(x), \quad 0 \le p \le 1$$

$F_1(x)$ is the cdf of a discrete r.v.,
$F_2(x)$ is the cdf of a continuous r.v.

Can verify $F_X(x)$ satisfies the properties of a c.d.f.