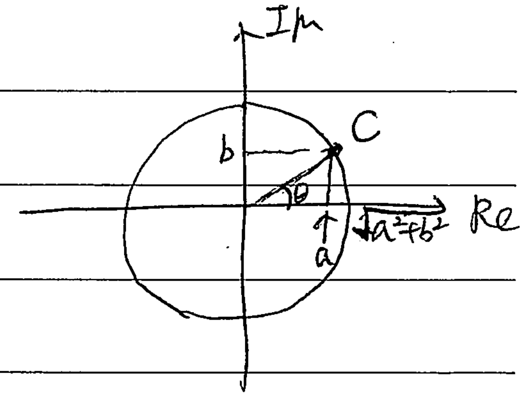
< Review of Polar Form >

Consider an arbitrary complex number $C = \underbrace{a}_{\text{Real}} + j\underbrace{b}_{\text{Imag.}}$

$\Rightarrow$ "$C = |C|\, e^{j\angle C}$"

$\text{mag}\{C\} = |C| = \sqrt{a^2 + b^2}$

$\text{phase}\{C\} = \underbrace{\cancel{\theta}\ \angle C = \cancel{\theta}\ \tan^{-1}\left(\frac{b}{a}\right)}_{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$



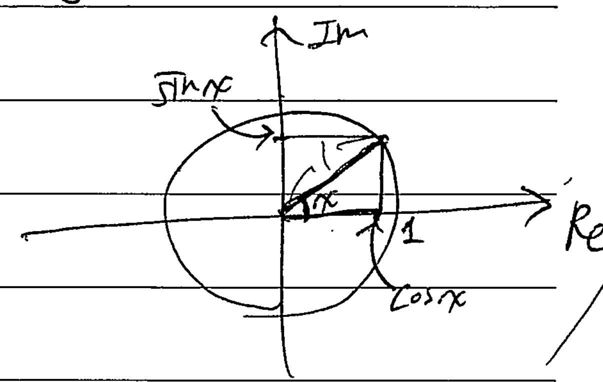
e.g. $e^{jx} = \underbrace{\cos x}_{\{Re\}} + j\underbrace{\sin x}_{\{Im\}}$

mag? $|e^{jx}| = \sqrt{\cos^2 x + \sin^2 x} = \sqrt{1} = 1$

$\text{slope} = \frac{\sin x}{\cos x} = \tan x$

$\text{angle} = x$

phase? $e^{j\angle x}$



(continue)

For arbitrary input $x[n]$, consider

$$x[n] \longrightarrow \boxed{\text{All-pass Filter}} \longrightarrow y[n]$$

Note : all-pass filter does not mean $y[n] = x[n]$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad\qquad E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \qquad\qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(\omega)|^2 d\omega$$

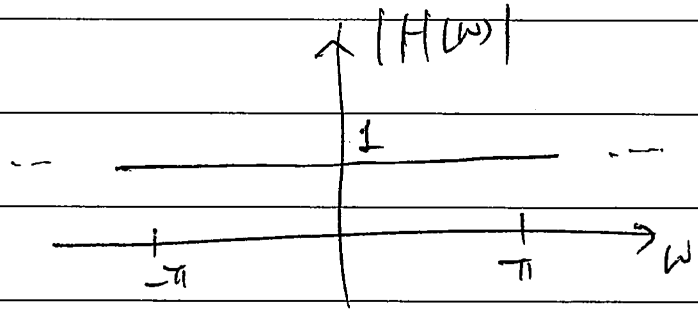
Since $Y(\omega) = H(\omega)\,X(\omega)$

$$|Y(\omega)| = \underbrace{|H(\omega)|}_{=1,\ \forall\omega}|X(\omega)|$$

$|Y(\omega)| = |X(\omega)|$ for all-pass filter.

Thus, $E_y = E_x$

$$H(\omega) = \frac{-a + e^{-j\omega}}{1 - a e^{-j\omega}} = \underbrace{\overbrace{e^{-j\omega}}^{e^{j\omega}\ \ -\omega}}_{1}\ \underbrace{\overbrace{\frac{1 - a e^{j\omega}}{1 - a e^{-j\omega}}}^{2\angle(1 - a e^{j\omega})}}_{1} \rightarrow \underline{|H(\omega)| = 1}$$



$$\angle H(\omega) = ? \quad -\omega + 2\angle(1 - a e^{j\omega})$$

$$1 - a e^{j\omega} = \underbrace{\{1 - a\cos\omega\}}_{Re} - \underbrace{j\, a \sin\omega}_{Im}$$

$$\angle(1 - a e^{j\omega}) = \underline{-\tan^{-1}\left\{\frac{a\sin\omega}{1 - a\cos\omega}\right\}}$$

Thus:

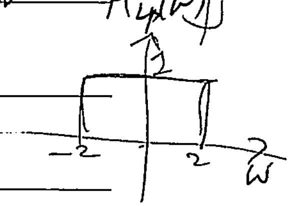
$$\angle H(\omega) = -\omega - 2\tan^{-1}\left\{\frac{a\sin\omega}{1 - a\cos\omega}\right\}$$

exercise). Consider the CT signal $x(t) = x_a(t) * \dfrac{\sin(2t)}{\pi t}$

where $x_a(t) = \sum_k \delta(t - 4\pi k)$. A DT signal is created

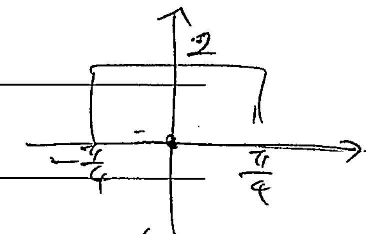
by sampling $x(t)$ accordingly to $x[n] = x(nT_s)$ for

$T_s = \dfrac{2\pi}{4}$. $y[n] = x[n] * h[n]$, $x[n] = x(nT_s)$

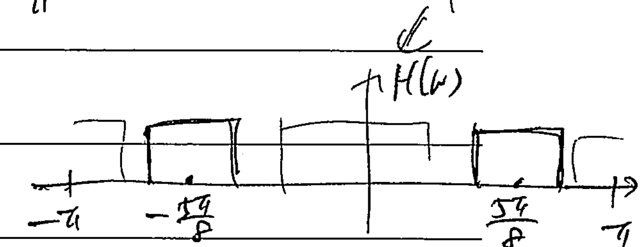


$x_a(t) \longrightarrow \boxed{h_{LP} = \dfrac{\sin(2t)}{\pi t}} \xrightarrow{x(t)} \boxed{\text{Ideal A/D} \quad \omega_s = \dfrac{2\pi}{T_s} = 4} \longrightarrow x[n]$

$x[n] \longrightarrow \boxed{h[n] = 2\dfrac{\sin(\frac{\pi}{4}n)}{\pi n} \cos\left(\frac{5\pi}{8}n\right)} \longrightarrow y[n]$



$y[n] = ?$ Plot $|Y(\omega)|$ over $-\pi < \omega < \pi$



⇒ Using FS expansion. (or you may use table 4.2)

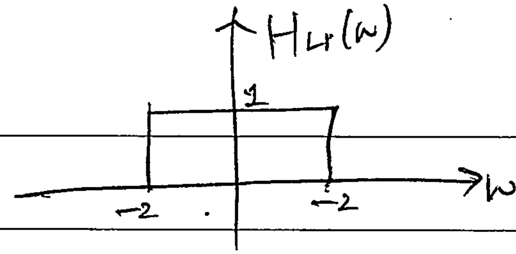
$$x_a(t) = \sum_{k=-\infty}^{\infty} \delta(t - 4\pi k) = \sum_{k=-\infty}^{\infty} \frac{1}{4\pi} e^{j\frac{2\pi k}{4\pi}t} \qquad \left(= \sum_k a_k e^{j\frac{2\pi k}{T}t} \right)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{4\pi} e^{j\frac{k}{2}t} \qquad \left(a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_a(t) e^{-j\frac{2\pi k}{T}t} dt \right)$$

Input freq: $\frac{k}{2}$, $-\infty < k < \infty$

$\cdots, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

$h_{LP}(t) \xleftrightarrow{DTFT}$



(Recall) $\left(x[n] = e^{jw_0 n} \longrightarrow \boxed{H(w)} \longrightarrow y[n] = H(w_0)\, e^{jw_0 n} \right)$

passes freq : $\underbrace{-2}_{k=-4},\ -\frac{3}{2},\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \underbrace{2}_{k=4}$ $\quad \frac{k}{2}$

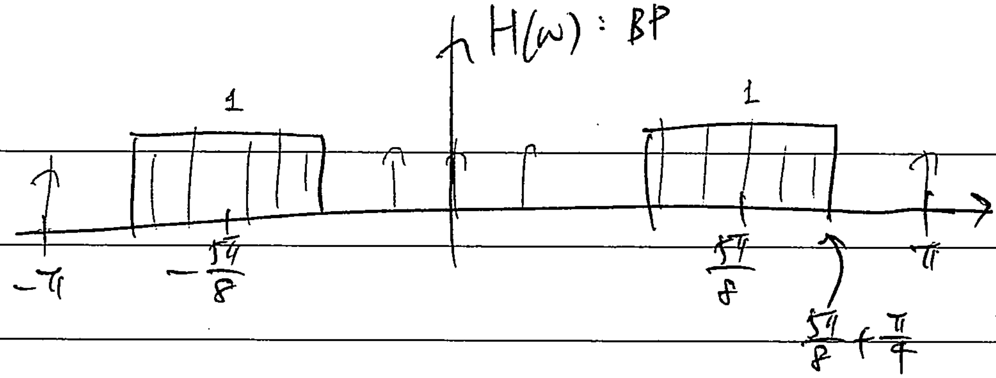
(Recall) (when we sample complex sinewave.

$$e^{jw_a t}\Big|_{t=nT_s} = e^{j(w_a T_s)\, n}$$

$$\text{DT freq} = T_s \times \underset{(\text{analog})}{\text{CT}}\ \text{freq}. \quad)$$

Thus, $x[n] = \frac{1}{4\pi} \sum_{k=-4}^{4} e^{j\frac{k}{\cancel{2}}\cdot\frac{\cancel{2}\pi}{4}n}$

$$= \frac{1}{4\pi} \sum_{k=-4}^{4} e^{jk\frac{\pi}{4}n} \Rightarrow \text{freq} : \left(-\pi,\ -\frac{3\pi}{4},\ -\frac{\pi}{2},\ -\frac{\pi}{4},\ 0,\ \pi,\ \frac{3\pi}{4},\ \frac{\pi}{2},\ \frac{\pi}{4}\right)$$

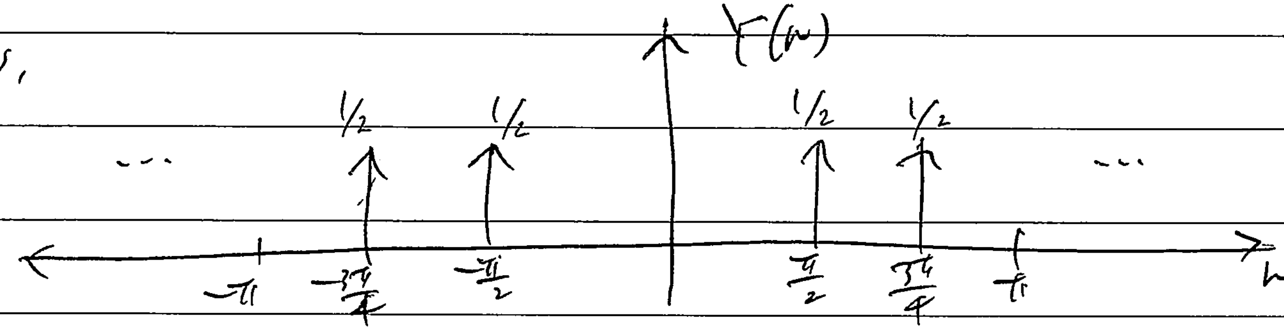


passes: $-\frac{3\pi}{4}, \; -\frac{\pi}{2}, \; \frac{\pi}{2}, \; \frac{3\pi}{4}$

$$y[n] = \frac{1}{4\pi}\left\{ e^{-j\frac{3\pi}{4}n} + e^{-j\frac{\pi}{2}n} + e^{j\frac{\pi}{2}n} + e^{j\frac{3\pi}{4}n} \right\}$$

(Recall) ($e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\,\delta(\omega-\omega_0)$, over $-\pi<\omega<\pi$, assume $-\pi<\omega_0<\pi$)

Thus,



※ Summary of Fourier Transform

Continuous-Time vs Discrete-Time.

CTFT : $X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt$

DTFT : $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

I-CTFT : $x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega) e^{j\omega t} d\omega$

I-DTFT : $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

Laplace T : $X_a(s) = \int_{-\infty}^{\infty} x_a(t) e^{-st} dt$

ZT : $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$X_a(\omega) = X_a(s) \Big|_{s=j\omega}$

for finite energy signals.

$X(\omega) = X(z) \Big|_{z=e^{j\omega}}$

for finite energy signals.