

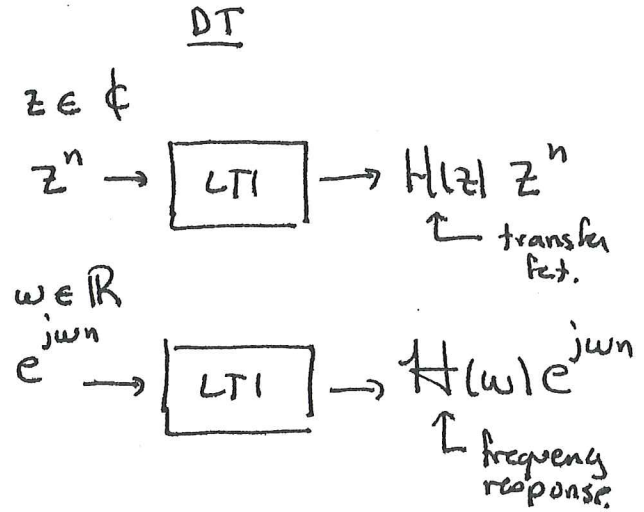
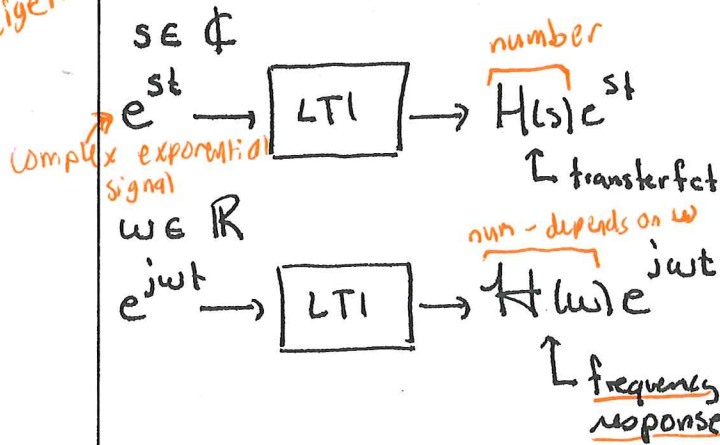
$H(s)$  = transfer function of the system

don't fail!

## 2. Response of LTI systems to $\mathbb{C}$ exponentials

eigen function

$s = \sigma + j\omega$   
 $\sigma$  number



Why?

In CT, let  $x(t) = e^{st}$

$$\text{then } y(t) = x(t) * h(t)$$

$$= h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= \underbrace{e^{st}}_{x(t)} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}_{H(s)}$$

special case  $s = j\omega$

$$\text{then } y(t) = e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}_{H(j\omega)}$$

Examples  $x(t) = e^t$ , here  $s=1$

$$x(t) = e^t \longrightarrow \boxed{\text{LTI } h(t)} \longrightarrow e^t H(1)$$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{t-\tau} d\tau = e^t \int_{-\infty}^{\infty} h(\tau) e^{-\tau} d\tau \end{aligned}$$

$$\underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-\tau} d\tau}_{\text{number}} = H(1)$$

eigen value of the eigen function  $e^t$

$$x(t) = e^{j2\pi f_0 t}$$

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{j2\pi f_0 (t-\tau)} d\tau$$

$$= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau} d\tau$$

Complex num,

↓

freq

number,  $H(j2\pi f_0) = H(2\pi f_0)$   
 transfer function      freq response

Why?

In DT, let  $x[n] = z^n$

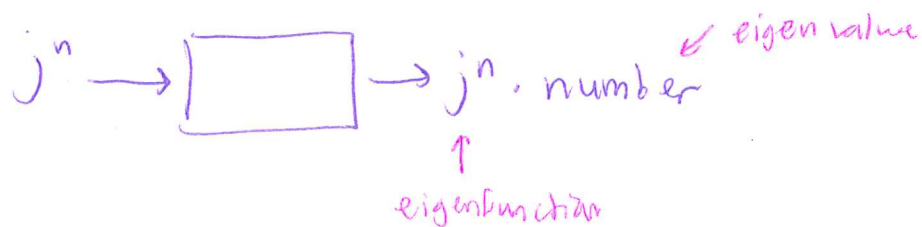
$$\begin{aligned} \text{then } y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\ &= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}_{H(z)} \end{aligned}$$

Special case  $z = e^{j\omega}$

$$\text{then } y[n] = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}_{H(\omega)}$$

Examples: Let  $x[n] = j^n$

$$\begin{aligned} \text{then } y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] j^{n-k} \\ &= j^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] j^{-k}}_{\text{a number}} \end{aligned}$$



### 3. Fourier series of CT periodic signals

Let  $x(t)$  be a periodic signal with period  $T$ .

Write  $\omega_0 = 2\pi/T$ .

Then the Fourier series of  $x(t)$  is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{where } a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

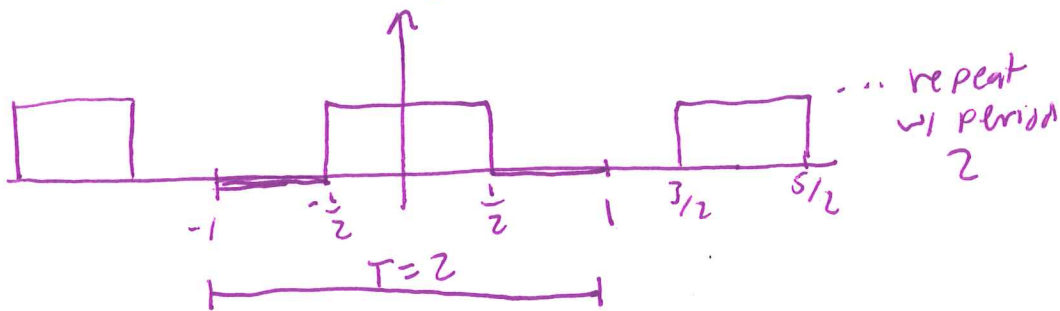
- dot product  
- if complex  
↳ conjugate

Examples:

Projection of  $x(t)$  onto  $e^{jk\omega_0 t}$

harmonic family  
all share period  $T$

↑ period



$$\left\{ e^{jk \frac{2\pi}{2} t} \right\}_{k \in \mathbb{Z}} = \left\{ e^{jk\pi t} \right\}$$

$$k=0 \quad a_0 = \frac{1}{2} \int_{-1}^1 x(t) e^{j0\pi t} dt$$

$$= \frac{1}{2} \int_{-1}^1 1 dt = \boxed{\frac{1}{2}}$$

average signal  
over 1 period

so

$$x(t) = \frac{1}{2} + \sum_{k \neq 0} \frac{\sin(k\pi/2)}{k\pi} e^{jk\pi t}$$

$$a_k = \frac{1}{2} \int_{-1}^1 x(t) e^{-jk\pi t} dt$$

$$= \frac{1}{2} \int_{-1}^1 e^{-jk\pi t} dt$$

$$= \frac{1}{2} \left. \frac{e^{-jk\pi t}}{-jk\pi} \right|_{-1}^1 = \frac{1}{2} \left[ \frac{e^{-jk\pi/2} - e^{jk\pi/2}}{-jk\pi} \right] = \frac{\sin(k\pi/2)}{k\pi}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

---

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

Ex]  $x(t) = 3 \cos 3t + (1+6j) \sin 6t$

$$= 3 \left( \frac{e^{3jt} + e^{-3jt}}{2} \right) + (1+6j) \left( \frac{e^{6jt} - e^{-6jt}}{2j} \right)$$

$$= \frac{3}{2} e^{3jt} + \frac{3}{2} e^{-3jt} + \left( \frac{1+6j}{2j} \right) e^{6jt} - \left( \frac{1+6j}{2j} \right) e^{-6jt} \quad (*)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ take } \omega_0 = 3 \rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk3t}$$

by comparison of ~~(\*)~~ and ~~(\*\*)~~

$$a_1 = \frac{3}{2} \quad a_2 = \frac{1+6j}{2j}$$

$$a_{-1} = \frac{3}{2} \quad a_{-2} = \frac{-(1+6j)}{2j}$$

$$a_0 = 0$$

$$\begin{aligned} &= \dots a_{-2} e^{-6jt} + a_{-1} e^{-3jt} \\ &\quad k=-2 \quad k=-1 \\ &+ \underline{a_0} + a_1 e^{3jt} + a_2 e^{6jt} \\ &\quad \uparrow \text{not in } (*) \\ &\quad s_0 = 0 \end{aligned}$$

# EX] DTFT of periodic signal

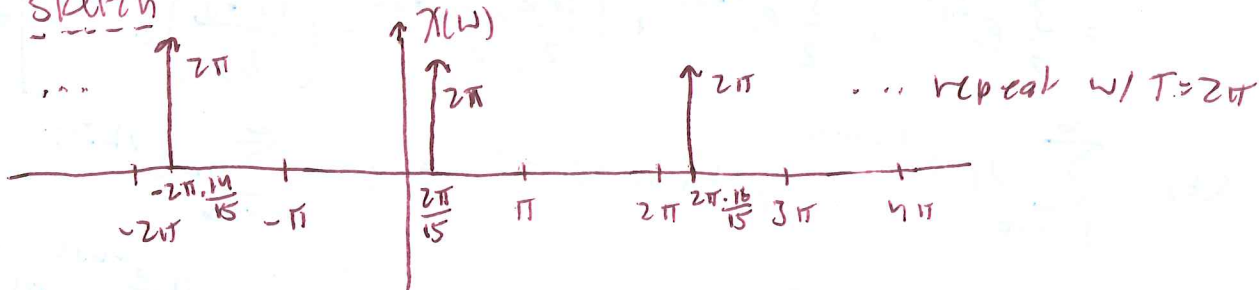
helps w / hw

①  $x[n] = e^{j\frac{2\pi}{15}n}$ , period  $N=15$

guess is  $X(\omega) = 2\pi \delta(\omega - \frac{2\pi}{15})$  if  $0 \leq \omega < 2\pi$

$X(\omega + 2\pi) = X(\omega)$ ,  $\forall \omega$  (repeats)

Sketch



$$\frac{2\pi}{15} - 2\pi = -2\pi \cdot \frac{14}{15}$$

$$\frac{2\pi}{15} + \frac{2\pi \cdot 15}{15} = 2\pi \frac{16}{15}$$

$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_0^{2\pi} 2\pi \delta(\omega - \frac{2\pi}{15}) e^{j\omega n} d\omega$$

$$= e^{j\frac{2\pi}{15}n}$$

$$= x[n] \checkmark$$

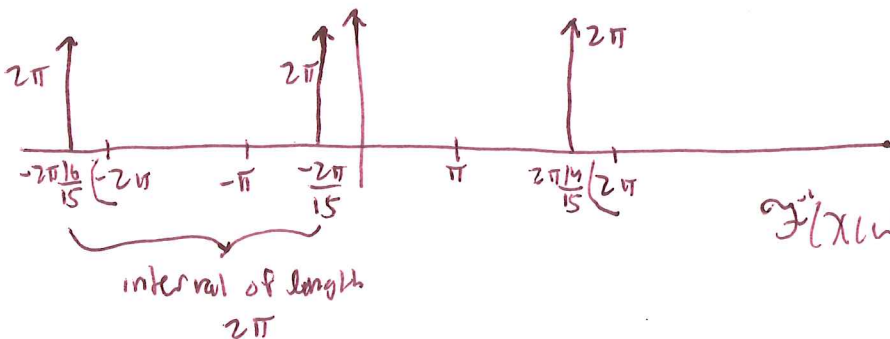
$\forall$  "for all"

$\exists$  "there exists"

②  $x[n] = e^{-j\frac{2\pi}{15}n}$  period  $N=15$

guess is  $X(\omega) = 2\pi \delta(\omega + \frac{2\pi}{15})$ , for  $-2\pi < \omega \leq 0$

$X(\omega + 2\pi) = X(\omega)$ ,  $\forall \omega$



$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-2\pi}^0 X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-2\pi}^0 2\pi \delta(\omega + \frac{2\pi}{15}) e^{j\omega n} d\omega$$

$$= e^{-j\frac{2\pi}{15}n}$$

$$= x[n] \checkmark$$

#### 4. Fourier Series of DT periodic signals

Let  $x[n]$  be a periodic signal with period  $N$ .

Then the Fourier series of  $x[n]$  is

$$\underline{x[n]} = \sum_{k=0}^{N-1} a_k e^{jk \frac{2\pi}{N} n} \quad \text{where} \quad \underline{a_k} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk \frac{2\pi}{N} n}$$

$n_0 + N - 1$   
 $\swarrow$  do not prod  
 one period  
 $n = n_0$

Why is the sum finite?

Because the set  $\left\{ e^{jk \frac{2\pi}{N} n} \right\}_{k \in \mathbb{Z}}$  of harmonically related exponentials only contains a finite number of distinct DT functions.

periodic DT signal

$[x[0], x[1], x[2], \dots, x[n-1]]$

periodic DT signal

$[a_0, a_1, a_2, \dots, a_{N-1}]$

$a_N = a_0$

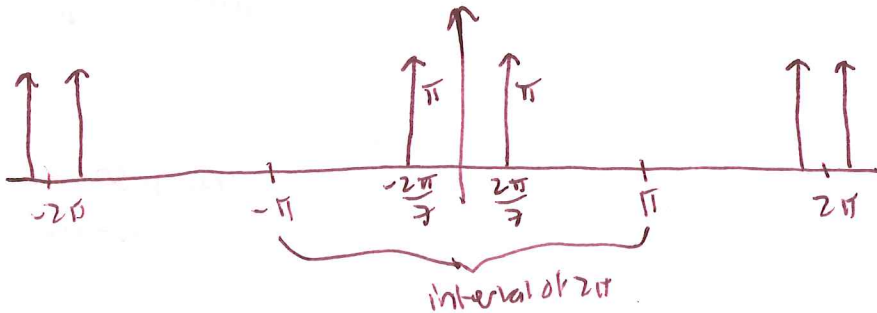
$a_{N+1} = a_1$



$$\textcircled{3} X[n] = \cos\left(\frac{2\pi}{7}n\right) \quad N=7$$

$$= \frac{e^{j\frac{2\pi}{7}n} + e^{-j\frac{2\pi}{7}n}}{2}$$

guess  $\rightarrow X(\omega) = \pi \delta\left(\omega - \frac{2\pi}{7}\right) + \pi \delta\left(\omega + \frac{2\pi}{7}\right)$ , for  $-\pi \leq \omega < \pi$



$$\tilde{f}(X(\omega)) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi \delta\left(\omega - \frac{2\pi}{7}\right) + \pi \delta\left(\omega + \frac{2\pi}{7}\right)) e^{j\omega n} d\omega$$

$$= \frac{e^{j\frac{2\pi}{7}n} + e^{-j\frac{2\pi}{7}n}}{2} = \cos\left(\frac{2\pi}{7}n\right)$$

\* (chou) tricks too! could specify on tests  $e^{jt}$

Examples:  $x[n] = \sin(3\pi n + \frac{\pi}{2}) = (-1)^n$

$N = \frac{2\pi}{3\pi} k \quad k=3 \rightarrow N=2$   
integer

$x[0] = \sin(\frac{\pi}{2}) = 1$

$x[1] = \sin(3\pi + \frac{\pi}{2})$

$x[2] = 1$

$x[3] = -1$

compute fourier coefficient

$a_0 = \frac{1}{2} \sum_{n=0}^1 x[n] e^0 = \frac{1}{2} \sum_{n=0}^1 x[n]$  (avg  $x[n]$  over 1 period)

$a_1 = \frac{1}{2} \sum_{n=0}^1 x[n] e^{-j\frac{\pi}{2}n} = \frac{1}{2} \sum_{n=0}^1 x[n] e^{-j\pi n}$

$= \frac{1}{2} x[0] e^0 + \frac{1}{2} x[1] e^{-j\pi}$

$= \frac{1}{2} (1)(1) + \frac{1}{2} (-1)(-1) = 1$

Ans

$a_0 = 0$

$a_1 = 1$

$a_{k+2} = a_k$ , for all other  $k$

(repeat periodically w/ period 2)

Quick method

$x[n] = (-1)^n \cong (e^{j\pi})^n = e^{j\pi n} + 0 + 0 e^{-j\pi n}$

compare w/ Fourier series formula

$x[n] = \sum_{k=0}^1 a_k e^{jk\pi n} = a_0 e^0 + a_1 e^{j\pi n} = \boxed{a_0 + a_1 e^{j\pi n}}$   
 $k=0, k=1$

$\Rightarrow a_0 = 0$

$a_1 = 1$

$a_2 = 0$

$a_3 = 1$

$a_4 = 0$

$a_5 = 1$

$a_k = \begin{cases} 0, & \text{if } k \text{ even} \\ 1, & \text{if } k \text{ odd} \end{cases}$

time domain

freq domain

$$a_k \longrightarrow a_k H(k\omega_0)$$
$$\sum a_k e^{jk \frac{2\pi}{T} t} \longrightarrow \sum a_k H(k\omega_0) e^{jk \frac{2\pi}{T} t}$$

## 5. Fourier Series and LTI systems

Recall  $\underbrace{e^{j\omega t}}_{\text{periodic}} \rightarrow \boxed{\text{LTI}} \rightarrow e^{j\omega t} H(\omega)$

$$\Rightarrow x(t) = \sum_k a_k e^{jk\omega_0 t} \rightarrow \boxed{\text{LTI}} \rightarrow \sum_k a_k e^{jk\omega_0 t} H(k\omega_0)$$

Example: if  $x(t) = \sin(2\pi 440t) + \sin(2\pi 880t)$

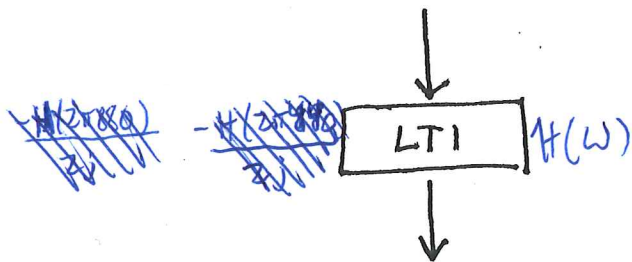
$T = \frac{1}{440}$        $T = \frac{1}{880} \Rightarrow 2T = \frac{1}{440}$  (freq. response (no j))

$$\sum_{k=-\infty}^{\infty} a_k e^{jk2\pi 440t} = \left\{ \begin{array}{l} \text{compare} \\ \leftarrow \omega \end{array} \right\} = \underbrace{\frac{1}{2j} e^{j2\pi 440t}}_{a_1} - \underbrace{\frac{1}{2j} e^{-j2\pi 440t}}_{a_{-1}} + \underbrace{\frac{1}{2j} e^{j2\pi 880t}}_{a_2} - \underbrace{\frac{1}{2j} e^{-j2\pi 880t}}_{a_{-2}}$$

\* share  $T = \frac{1}{440}$

$x(t)$  can be represented by sequence of  $a_k$ 's:

$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
0	$-\frac{1}{2j}$	$-\frac{1}{2j}$	0	$\frac{1}{2j}$	$\frac{1}{2j}$	0	0 ...



new sequence of  $a_k$ 's.

$$\frac{-H(2\pi 880)}{2j} \quad \frac{-H(2\pi 440)}{2j} \quad 0 \quad \frac{1}{2j} H(2\pi 440) \quad \frac{1}{2j} H(2\pi 880) \quad 0 \quad 0$$

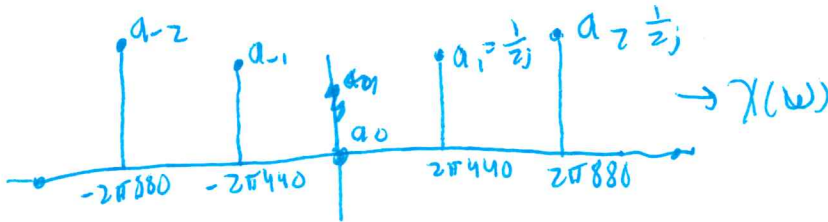
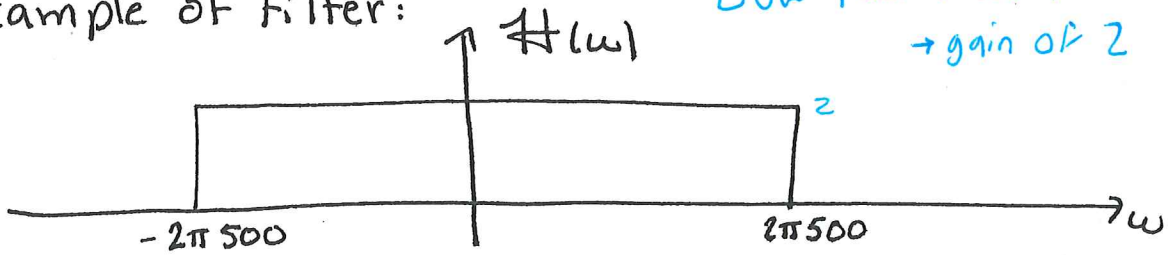
Output is

$$y(t) = -\frac{1}{2j} H(-2\pi 880) e^{-j2\pi 880t} - \frac{1}{2j} H(-2\pi 440) e^{-j2\pi 440t} + \frac{1}{2j} H(2\pi 440) e^{j2\pi 440t} + \frac{1}{2j} H(2\pi 880) e^{j2\pi 880t}$$

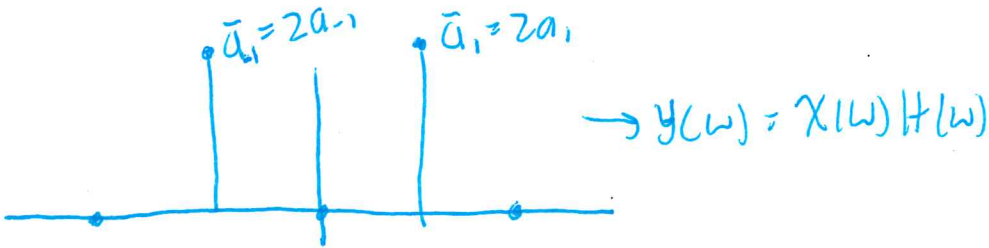
$$a_k \rightarrow \boxed{\text{LTI}} \rightarrow \bar{a}_k = a_k H(k\omega_0)$$

Example of filter:

Low Pass Filter  
→ gain of 2



\* outside of bands disappear when multiplied  
- inside gets multiplied by the gain



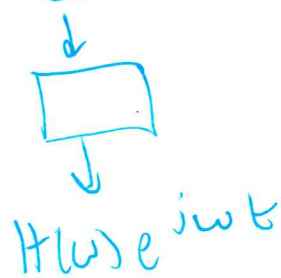
\* if input signal periodic w/ period  $\frac{1}{f_0}$

→ output signal also period  $\frac{1}{f_0}$ , but w/ fewer harmonics

$$\text{new } a_k = \text{old } a_k \cdot H(k \frac{2\pi}{T})$$

$$\text{here } H(\omega) = \begin{cases} 1, & |\omega| < 2\pi 500 \\ 0, & \text{else} \end{cases}$$

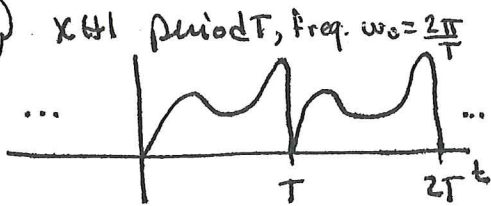
$x(t)$  = linear combo of  $e^{j\omega t}$ ,  $\omega \in \mathbb{R}$



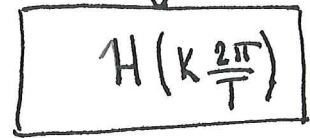
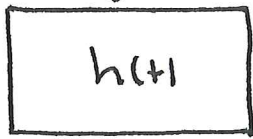
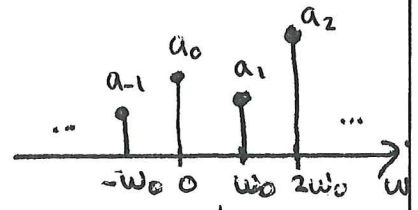
$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

# 1. Why Fourier Transforms?

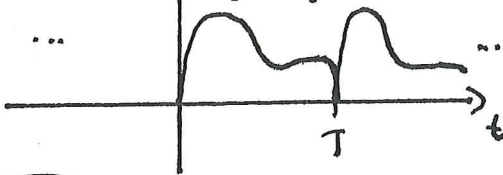
Recall



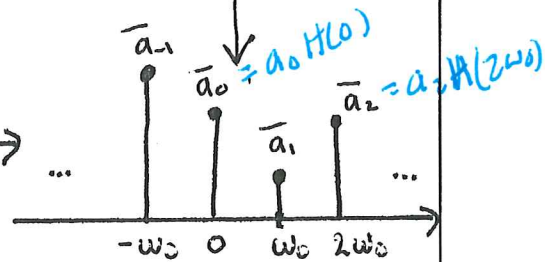
Fourier series



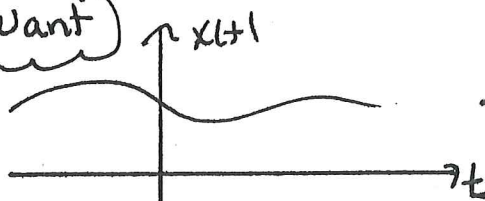
$y(t) = x(t) * h(t)$   
 $y(t)$  period  $T$  \*still repeats



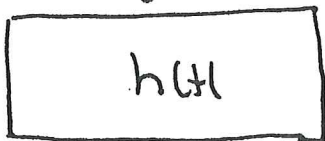
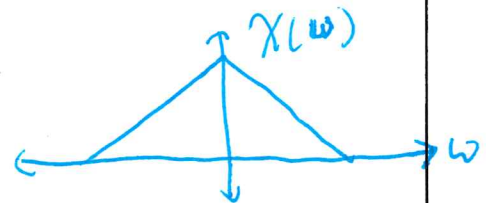
Fourier series



Now want

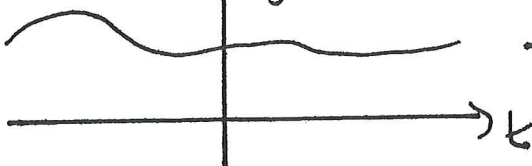


Fourier transform

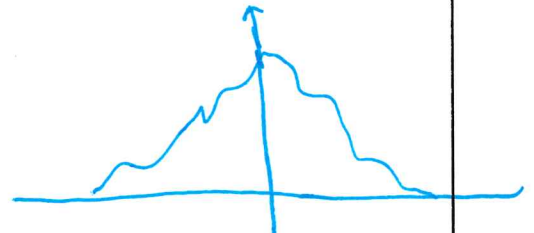


$y(t) = x(t) * h(t)$

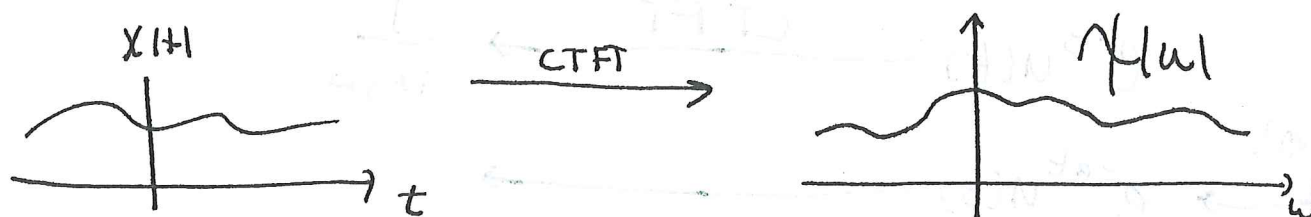
Fourier transform



$X(\omega)H(\omega) = Y(\omega)$



## 2. CT Fourier Transform (CTFT): Definition and inverse



$X(\omega)$  is the projection of the signal  $x(t)$  onto the signal  $e^{j\omega t}$

Formula CTFT of $x(t)$	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \mathcal{F}\{x(t)\}$
---------------------------	--

Intuition for formula:

recall  $H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$

\* freq. response is the Fourier transform of  $h(t)$   
(of LTI)

$$e^{j\omega t} \rightarrow \boxed{\phantom{H(\omega) e^{j\omega t}}} \rightarrow H(\omega) e^{j\omega t}$$

obs  $\rightarrow H(\omega) = \text{CTFT of } h(t)$



$$e^{-t} u(t) \xrightarrow{\text{CTFT}} \frac{1}{1+s}$$

$\{w\}$   
\*  $\rightarrow$

$$e^{-at} u(t) \xrightarrow{\hspace{10em}}$$

$a > 0$

$$f(t) \xrightarrow{\text{CTFT}} |$$

$$f(t-t_0) \xrightarrow{\hspace{10em}}$$

Example 1: Compute the CTFT  $X(j\omega)$  of the signal

$$x(t) = e^{-t} u(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-t} u(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-t} e^{-j\omega t} dt = \int_0^{\infty} e^{-t(1+j\omega)} dt$$

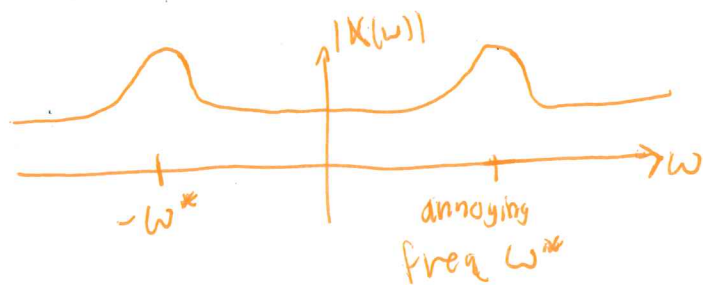
$$= \left. \frac{e^{-t(1+j\omega)}}{-(1+j\omega)} \right|_0^{\infty}$$

$$= 0 - \frac{-1}{1+j\omega}$$

$$= \frac{1}{1+j\omega}$$

$A_k$  is how much of freq  $k\omega_0$   
there is in the signal  $x(t)/x(n)$   
how much of " $e^{jk\omega_0 t} / e^{jk\omega_0 n}$ "

$X(\omega)$  is how much frequency  $\omega$  there is  
in the signal  $x(t)$   
how much of  $e^{j\omega t}$



Example 2: Compute the CTFT  $X(j\omega)$  of the signal  
 $x(t) = \delta(t)$ .

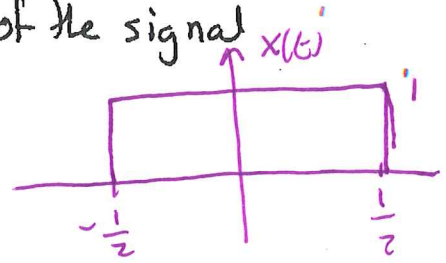
$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega(0)} dt \\ &= \int_{-\infty}^{\infty} \delta(t) \cdot 1 dt \\ &= \int_{-\infty}^{\infty} \delta(t) dt \\ &= 1 \end{aligned}$$

or  
"by sifting property"

$$\begin{aligned} &\delta(t-\tau) \\ &\vdots \\ &e^{-j\omega(t-\tau)} \end{aligned}$$

Example 3: Compute the CTFT  $X(\omega)$  of the signal

$$x(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{else} \end{cases}$$



"rect"

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-\frac{1}{2}}^{\frac{1}{2}}, \omega \neq 0$$

\* Cont ÷ by 0

$$= 1, \omega = 0$$

$$= \frac{e^{-j\omega \frac{1}{2}} - e^{j\omega \frac{1}{2}}}{-j\omega}, \omega \neq 0$$

$$= 1, \omega = 0$$

$$= \frac{2 \sin\left(\frac{\omega}{2}\right)}{\omega}, \omega \neq 0$$

$$= 1, \omega = 0$$

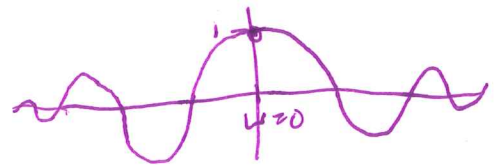
"sinc"

Euler's

L'Hopital's rule

$$\lim_{\omega \rightarrow 0} \frac{2 \sin\left(\frac{\omega}{2}\right)}{\omega} = \lim_{\omega \rightarrow 0} \frac{\cos\left(\frac{\omega}{2}\right)}{1} = 1$$

"sinc"



FT of a "rect" is a "sinc"

How to recover  $x(t)$  from  $X(\omega)$

Inverse  
CTFT  
formula

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \mathcal{F}^{-1}\{X(\omega)\}$$

weight of component  
w/ freq  $\omega$   
 $e^{j\omega t}$

Intuition behind formula:

$$\frac{X(\omega)}{2\pi} \sim a_k \text{ in Fourier series}$$

weight of component  $e^{j\omega t}$  in the signal

$$\left| \frac{X(\omega^*)}{2\pi} \right| \text{ Large!}$$

$$\text{CTFT} \left( \frac{1}{2\pi} \right) \Rightarrow \delta(\omega)$$

$$\frac{1}{2\pi} \xrightarrow{\text{CTFT}} \delta(\omega)$$
$$\delta(\omega) \xrightarrow{1 \text{ CTFT}} \frac{1}{2\pi}$$

Example 1: Compute the inverse CTFT of  $X(\omega) = \delta(\omega)$ .

$$\begin{aligned}
 x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega \\
 &= \frac{1}{2\pi} e^{j\omega t} \Big|_{\omega=0}, \text{ by sifting property of } \delta(\omega) \\
 &= \frac{1}{2\pi}
 \end{aligned}$$



T.S.

$$x(t-b_0) \xrightarrow{\text{CTFT}} e^{-j\omega b_0} X(\omega)$$

$$|e^{-j\omega b_0} X(\omega)| = |X(\omega)|$$

same frequency no matter  
when you "start recording"  
→ doesn't change magnitude

---

$$X^*(t) \xrightarrow{\text{CTFT}} X^*(-\omega)$$

general rule  
 $(f(\omega))^* \rightarrow$  just replace all  $j$ 's  
w/  $-j$ 's

# \* Know proofs for EXAM 2 $\mathcal{F} = \text{Fourier TF}$

## 3. Properties of CTFT

### Linearity

$$\mathcal{F}(ax(t) + by(t)) = a\mathcal{F}(x(t)) + b\mathcal{F}(y(t))$$

for any  $a, b \in \mathbb{C}$

\*Proof:

$$\int_{-\infty}^{\infty} (ax(t) + by(t))e^{-j\omega t} dt = a \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t)e^{-j\omega t} dt \quad \square$$

### Time-shifting

$$\mathcal{F}(x(t-t_0)) = e^{-j\omega t_0} \mathcal{F}(x(t))$$

\*Proof:  $\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)} d\tau$  (for any  $t_0 \in \mathbb{R}$  let  $\tau = t - t_0$   $d\tau = dt$ )

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega \tau} d\tau = e^{-j\omega t_0} \mathcal{F}(x(t)) \quad \square$$

### Conjugation and conjugate symmetry

If  $\mathcal{F}(x(t)) = |X(\omega)|$ , then  $\mathcal{F}(x(t)^*) = |X^*(-\omega)|$ .

\*Proof:  $|X^*(-\omega)| = \left( \int_{-\infty}^{\infty} x(t)e^{+j\omega t} dt \right)^*$  (conjugate everything inside)

$$= \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t} dt = \mathcal{F}(x(t)^*) \quad \square$$

This property implies that

- The real part of the F.T. of a real signal is even.
- The imaginary part of the F.T. of a real signal is odd.

Also

- The F.T. of a real and even signal is real and even.

- chose  $x(-t)$  so that  $x(t) = x(-t)$   $\rightarrow$  reflect over y-axis  
 \*make it even!  
 Part II-B  $X(t)$  real + even then  $X(\omega)$  is also real + even

why? because if  $x(t)$  is real

$$\text{then } x^*(t) = x(t)$$

$$\Rightarrow \mathcal{F}(x^*(t)) = \mathcal{F}(x(t))$$

$$\Rightarrow X^*(-\omega) = X(\omega), \text{ by conj. property}$$

$$\Rightarrow \operatorname{Re}(X(\omega)) - j \operatorname{Im}(X(\omega)) = \operatorname{Re}(X(\omega)) + j \operatorname{Im}(X(\omega))$$

$$\Rightarrow \operatorname{Re} X(-\omega) = \operatorname{Re}(X(\omega)) \quad \text{even}$$

$$- \operatorname{Im}(X(-\omega)) = \operatorname{Im}(X(\omega)) \quad \text{odd} \quad \square$$

If, in addition,  $x(t)$  is even

$$\Rightarrow x(-t) = x(t).$$

and therefore

$$\begin{aligned} X(-\omega) &= \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(-\tau) e^{-j\omega \tau} d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \tau} d\tau \\ &= X(\omega) \end{aligned}$$

so  $X(\omega)$  is even

$$\text{but } \underbrace{X(\omega)}_{\text{even}} = \underbrace{\operatorname{Re}(X(\omega))}_{\text{even}} + j \underbrace{\operatorname{Im}(X(\omega))}_{\text{odd}}$$

so odd part must be zero.

$$\Rightarrow X(\omega) \text{ real.} \quad \square$$

\* could be asked to prove on exam!

**Differentiation and integration** If  $\mathcal{F}\{x(t)\} = X(\omega)$

(1)  $\frac{d}{dt} x(t) \xrightarrow{\text{CTFT}} j\omega X(\omega)$  \* used later!

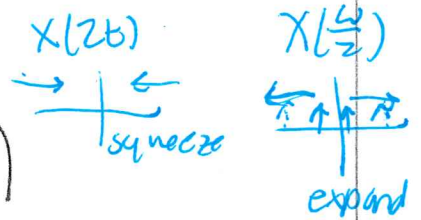
(2)  $\int_{-\infty}^t x(\tau) d\tau \xrightarrow{\text{CTFT}} \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$

**Time scaling**

If  $\mathcal{F}\{x(t)\} = X(\omega)$

then

$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$   
 $a \neq 0$



Proof:  $\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$  let  $\tau = at$

$$= \begin{cases} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{a}} \frac{d\tau}{a}, & a > 0 \\ \int_{\infty}^{-\infty} x(\tau) e^{-j\omega \frac{\tau}{a}} \frac{d\tau}{a}, & a < 0 \end{cases} = \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{a}} d\tau, & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{a}} d\tau, & a < 0 \end{cases}$$

**Frequency Scaling**

$X\left(\frac{\omega}{b}\right) \xrightarrow{\text{CTFT}} \frac{1}{|b|} x\left(\frac{t}{b}\right)$   
 $b \neq 0$

Proof let  $b = \frac{1}{a}$  in scaling property

"Therefore"  $x(at) \xrightarrow{\text{CTFT}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$\Rightarrow x\left(\frac{t}{b}\right) \xrightarrow{\text{CTFT}} |b| X(b\omega)$

by linearity, multiply both sides by  $\frac{1}{|b|}$

$\Rightarrow |b| x\left(\frac{t}{b}\right) \xrightarrow{\text{CTFT}} X(b\omega)$   
 $\square$

(Duality) If  $X(t) \xrightarrow{\text{CTFT}} X(\omega)$   
 then  $X(\omega) \xrightarrow{\text{CTFT}} 2\pi X(-\omega)$

Proof:

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \text{ let } \tau = \omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau) e^{j\tau t} d\tau$$

$$\Rightarrow X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau) e^{j\tau \omega} d\tau$$

$$\Rightarrow 2\pi X(-\omega) = \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} X(\tau) e^{-j\tau \omega} d\tau$$

let  $t = \tau$

$$= \int_{-\infty}^{\infty} \underbrace{X(t)}_{\text{"blah"}} e^{-j t \omega} dt = \text{CTFT}(X(t)) \quad \square$$

# Parseval's Relation

$E_{\infty}$   
Total energy  
// of  $x(t)$

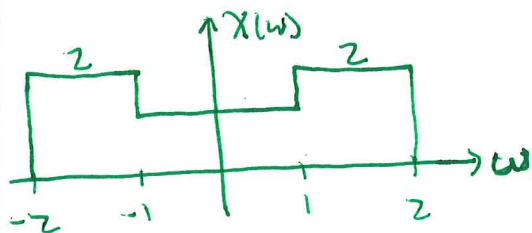
energy of  
CTFT  $X(\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Ex Q.1

A signal  $x(t)$  has CTFT given by following graph

Does  $x(t)$  have finite energy?



Ans)  $E_{\infty} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$ , by Parseval's Relation

$$= \frac{1}{2\pi} \left( \int_{-2}^{-1} 2^2 d\omega + \int_{-1}^1 1^2 d\omega + \int_1^2 2^2 d\omega \right)$$

$$= \frac{1}{2\pi} (4 + 2 + 4)$$

$$= \frac{5}{\pi}$$

so  $E_{\infty}$  is finite

\* can use if asked to evaluate difficult integral

$$\int_{-\infty}^{\infty} \sin^2 t dt = \int_{-\infty}^{\infty} |\text{CTFT}(\sin t)|^2 dt$$

↑ two direct deltas

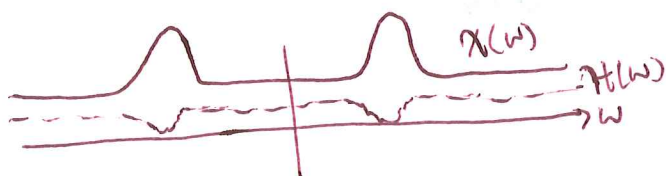
recall an LTI system

$$x(t) \rightarrow \boxed{h(t)} \rightarrow x(t) * h(t) = y(t)$$

$$x(t) = \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega$$

$$y(t) = \frac{1}{2\pi} \int Y(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int X(\omega) H(\omega) e^{j\omega t} d\omega$$



## Convolution Property

$$x_1(t) * x_2(t) \xrightarrow{\text{CTFT}} X_1(\omega) X_2(\omega)$$

So for LTI systems, the CTFT of the output  $y(t)$  is

$$Y(\omega) = X(\omega) H(\omega)$$

↑ frequency response.

and

$$y(t) = \mathcal{F}^{-1} ( X(\omega) H(\omega) )$$

Recall Properties of LTI systems

in time domain

①  $x(t) \rightarrow \boxed{h(t)} \rightarrow x(t) * h(t)$

②  $x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t)$

same as

$$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$$

in frequency domain

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow X(\omega) H(\omega) = Y(\omega)$$

$$X(\omega) \rightarrow \boxed{H_1(\omega)} \rightarrow \boxed{H_2(\omega)} \rightarrow Y(\omega)$$

$$X(\omega) \rightarrow \boxed{H_1(\omega) H_2(\omega)} \rightarrow Y(\omega)$$



## multiplication property

$$x_1(t) x_2(t) \xrightarrow{\text{CTFT}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Example:

$$x(t) e^{j\omega_0 t} \xrightarrow{\text{CTFT}} \frac{1}{2\pi} X(\omega) * 2\pi \delta(\omega - \omega_0) = X(\omega) * \delta(\omega - \omega_0)$$

phase shift freq. shift

#### 4. Fourier transform of CT periodic signals

If signal periodic with period  $T$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

What is the Fourier transform of this Fourier series?

$$\begin{aligned} \mathcal{F}\{x(t)\} &= \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\} \\ &= \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\{e^{jk\omega_0 t}\}, \quad \text{by linearity of CFT} \end{aligned}$$

So all we need to know is  $\mathcal{F}\{e^{j\omega_0 t}\}$ ,  $\omega_0 \in \mathbb{R}$ .

but,

~~$$\mathcal{F}\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt$$~~

~~$$= \int_{-\infty}^{\infty} e^{jt(k\omega_0 - \omega)} dt$$~~

~~don't try to CFT a periodic signal!~~

~~$$= \begin{cases} \frac{e^{jt(k\omega_0 - \omega)}}{j(k\omega_0 - \omega)}, & \omega_0 \neq \omega \\ \int_{-\infty}^{\infty} 1 dt, & \omega_0 = \omega \end{cases} \quad \infty$$~~

- function must "taper down" to 0 in order to integrate

CFT of  $x(t) = 1$

$$\int_{-\infty}^{\infty} 1 e^{-j\omega t} dt$$

guess  $\mathcal{F}(e^{j\omega_0 t}) = 2\pi \delta(\omega - \omega_0)$

check:  $\mathcal{F}^{-1}(2\pi \delta(\omega - \omega_0)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega$

$$= \int_{-\infty}^{\infty} e^{j\omega t} \delta(\omega - \omega_0) d\omega$$

$$= e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega$$

$$= e^{j\omega_0 t}$$

so

$e^{j\omega_0 t} \xrightarrow{\text{CTFT}} 2\pi \delta(\omega - \omega_0)$
--

$$X(t) = e^{j\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{e^{j\omega t}}{2\pi \delta(\omega - \omega_0)} d\omega$$

$$\sin \omega_0 t \xrightarrow{\text{CTFT}} \frac{2\pi \delta(\omega - \omega_0) - 2\pi \delta(\omega + \omega_0)}{2j} = \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0)$$

$$\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\cos \omega_0 t \longrightarrow \pi (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$

## 5. Frequency Response of <sup>CT</sup>LTI systems

Recall: for LTI systems

$$e^{j\omega t} \rightarrow \boxed{h(t)} \rightarrow e^{j\omega t} H(\omega)$$

$$\text{where } H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$H(\omega)$  is the CTFT of  $h(t)$

Now we know that, more generally

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \rightarrow \boxed{h(t)} \rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega$$

or, viewing system in frequency domain

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) = X(\omega) H(\omega)$$

Example:

$$x(t) = 1$$

$$h(t) = \delta(t - t_0)$$

$$y(\omega) = \mathcal{F}\{x(t)\} \mathcal{F}\{h(t)\}$$

$$= \mathcal{F}\{1\} \mathcal{F}\{\delta(t - t_0)\}$$

$$= 2\pi \delta(\omega) \cdot e^{-j\omega t_0} \mathcal{F}\{\delta(t)\}$$

$$= 2\pi \delta(\omega) e^{-j\omega t_0} \cdot 1$$

$$= 2\pi \underline{e^{j\omega t_0}} \delta(\omega)$$

$$= 2\pi \delta(\omega)$$

↓ ICTFT

↑

Same as:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau - t_0) d\tau = 1$$

## \* Probs on test!

One can define an LTI system using constant coefficient differential equations

$$\sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t) = \sum_{m=0}^M b_m \frac{d^m}{dt^m} x(t)$$

Examples:  $H(\omega) = ?$

We will use  $y(\omega) = X(\omega)H(\omega)$

$$+ \frac{d}{dt} x(t) \xrightarrow{\text{CTFT}} j\omega X(\omega)$$

$$\frac{d}{dt} \left( \frac{d}{dt} x(t) \right) \xrightarrow{\text{CTFT}} j\omega (j\omega X(\omega))$$

EX1

$$12 \frac{d}{dt} y(t) + 7y(t) = x(t)$$

[CTFT of both sides]

$$12j\omega Y(\omega) + 7Y(\omega) = X(\omega)$$

$$Y(\omega)(12j\omega + 7) = X(\omega)$$

$$Y(\omega) = X(\omega) \left( \frac{1}{12j\omega + 7} \right) = H(\omega) = \frac{1}{12j\omega + 7}$$

Trick to obtain frequency response of CT LTI systems defined by a differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^M b_m \frac{d^m x(t)}{dt^m}$$

$$\Rightarrow \mathcal{F} \left( \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} \right) = \mathcal{F} \left( \sum_{m=0}^M b_m \frac{d^m x(t)}{dt^m} \right)$$

$$\Rightarrow \sum_{k=0}^N a_k \mathcal{F} \left( \frac{d^k y(t)}{dt^k} \right) = \sum_{m=0}^M b_m \mathcal{F} \left( \frac{d^m x(t)}{dt^m} \right)$$

$$\Rightarrow \sum_{k=0}^N a_k (j\omega)^k Y(\omega) = \sum_{m=0}^M b_m (j\omega)^m X(\omega)$$

$$\Rightarrow Y(\omega) = \frac{\sum_{m=0}^M b_m (j\omega)^m}{\underbrace{\sum_{k=0}^N a_k (j\omega)^k}_{H(\omega)}} X(\omega)$$

So

$$H(\omega) = \frac{\sum_{m=0}^M b_m (j\omega)^m}{\sum_{k=0}^N a_k (j\omega)^k}$$



# 1. DT Fourier transform: definition and inverse

$$x[n] \xrightarrow{\text{DTFT}} \mathcal{X}(\omega)$$

$$\mathcal{X}(\omega) \xrightarrow{\text{inverse DTFT}} x[n]$$

Formulas

$$\begin{array}{l} \text{DTFT} \quad \mathcal{X}(\omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ \text{IDTFT} \quad x[n] = \mathcal{F}^{-1}\{\mathcal{X}(\omega)\} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{X}(\omega) e^{j\omega n} d\omega \end{array}$$

$\omega \in \mathbb{R}$   
periodic w/  $2\pi$   
has to have length  $2\pi$

Example:

$$x[n] = 3^{-n} u[n]$$

$$\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} 3^{-n} u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} 3^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{3e^{j\omega}}\right)^n$$

since  $\left|\frac{1}{3e^{j\omega}}\right| = \frac{1}{3} < 1$

$$\Rightarrow \mathcal{X}(\omega) = \frac{1}{1 - \frac{1}{3e^{j\omega}}} = \frac{3e^{j\omega}}{3e^{j\omega} - 1}$$

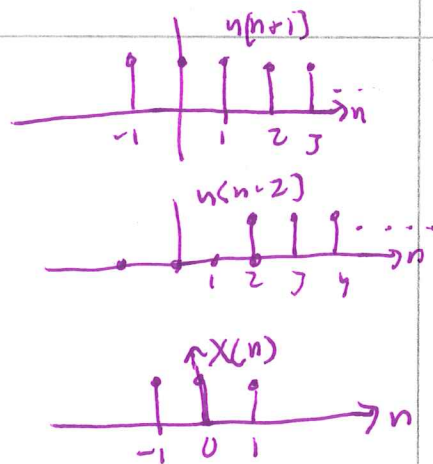
example:  $x[n] = u[n+1] - u[n-2]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-1}^1 e^{-j\omega n}$$

$$= e^{j\omega} + e^0 + e^{-j\omega}$$

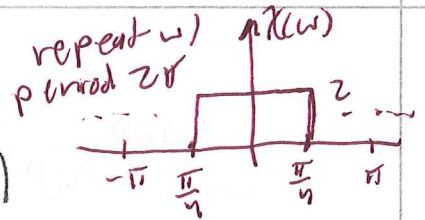
$$= 1 + 2 \cos \omega$$



## 2. Properties of the DTFT

periodicity

$$X(\omega + 2\pi) = X(\omega)$$



Proof:

$$\begin{aligned} X(\omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + 2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{=1} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(\omega) \quad \checkmark \end{aligned}$$

Linearity

$$a x_1[n] + b x_2[n] \xrightarrow{\text{DTFT}} a X_1(\omega) + b X_2(\omega)$$

Proof:

$$\begin{aligned} \mathcal{F}(a x_1[n] + b x_2[n]) &= \sum_{n=-\infty}^{\infty} (a x_1[n] + b x_2[n]) e^{j\omega n} \\ &= a \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2[n] e^{-j\omega n} \\ &= a \mathcal{F}(x_1[n]) + b \mathcal{F}(x_2[n]). \quad \square \end{aligned}$$

$$e^{j\omega_0 n} x[n] \longrightarrow X(\omega - \omega_0)$$

$$\begin{aligned} \mathcal{F}(e^{j\omega_0 n} x[n]) &= \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega - \omega_0)n} \\ &= X(\omega - \omega_0) \quad \checkmark \end{aligned}$$

## time shifting and frequency shifting

$$X[n-n_0] \xrightarrow{\text{DTFT}} e^{-j\omega n_0} X(\omega)$$

$$e^{j\omega_0 n} X[n] \xrightarrow{\text{DTFT}} X(\omega - \omega_0)$$

Proof:

$$F\{X[n-n_0]\} = \sum_{n=-\infty}^{\infty} \underbrace{X[n-n_0]}_{=m} e^{-j\omega n}$$

$$\text{let } m = n - n_0$$

$$= \sum_{m=-\infty}^{\infty} X[m] e^{-j\omega(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} X[m] e^{-j\omega m} \underbrace{e^{-j\omega n_0}}$$

$$= e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} X[m] e^{-j\omega m}$$

$$= e^{-j\omega n_0} X(\omega)$$



note:

$$|e^{-j\omega n_0} X(\omega)| = |X(\omega)|$$

hw

$$X(\omega) = \sin \omega$$

$$= \underbrace{\sin(\omega)}_{\text{odd}} + j \underbrace{0}_{\substack{\text{even} \\ \text{odd}}}$$

so signal  $X[n]$  not real

\*helps w/ hw?

# Conjugation and conjugate symmetry

$$x^*[n] \xrightarrow{\text{DTFT}} X^*(-\omega)$$

Proof: if  $x[n]$  real

$$\text{i.e. } x^*[n] = x[n]$$

$$\Rightarrow X^*(-\omega) = X(\omega)$$

$$\underbrace{\text{Re}(-\omega)}_{x[n] \text{ real}} - j \underbrace{\text{Im}(-\omega)}_{y[n] \text{ pure imag.}} = \underbrace{\text{Re}(\omega)}_{x[n] \text{ real}} + j \underbrace{\text{Im}(\omega)}_{y[n] \text{ pure imag.}}$$

multiply both sides by  $j$

$$\begin{aligned} \text{Re}(-\omega) = \text{Re}(\omega) &\leftarrow \text{Re}(\omega) \text{ even} \\ -\text{Im}(-\omega) = \text{Im}(\omega) &\leftarrow \text{Im}(\omega) \text{ odd} \end{aligned}$$

$$\begin{aligned} \text{Re}(\omega) &\text{ odd} \\ \text{Im}(\omega) &\text{ even} \end{aligned}$$

$$\begin{aligned} j \text{Re}(-\omega) &= j \text{Re}(\omega) \\ -\text{Im}(-\omega) &= \text{Im}(\omega) \end{aligned}$$

Important Corollary: if  $x[n]$  is real, then  $|X(\omega)| = |X^*(-\omega)|$ .

$\Rightarrow \text{Re}(X(\omega))$  even function of  $\omega$   
 $\text{Im}(X(\omega))$  odd function of  $\omega$

# Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(\omega)|^2 d\omega$$

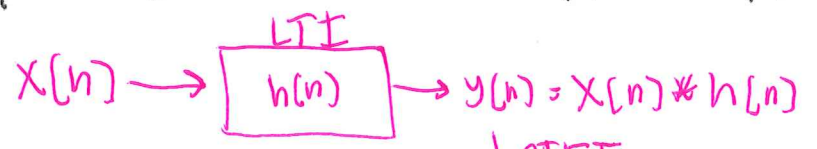
energy of signal

$$\int_0^{2\pi} |X(\omega)|^2 d\omega$$

energy of F.T.

# Convolution Property

$$x_1[n] * x_2[n] \xrightarrow{\text{DTFT}} X_1(\omega) X_2(\omega)$$



$$X(\omega) \qquad \qquad \qquad \downarrow \text{DTFT} \qquad \qquad \qquad Y(\omega) = X(\omega)H(\omega)$$



## Multiplication Property

$$x[n] y[n] \xrightarrow{\text{DTFT}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

No duality in DT!

## Differentiation in frequency

$$n x[n] \xrightarrow{\text{DTFT}} j \frac{d}{d\omega} X(\omega).$$

### 3. Fourier transform of DT periodic signals

$$x[n] \text{ periodic } \Rightarrow x[n] = \sum_{k=0}^{N-1} a_k e^{jk \frac{2\pi}{N} n}$$

period  $N$

$$\begin{aligned} \Rightarrow \mathcal{F}(x[n]) &= \mathcal{F}\left(\sum_{k=0}^{N-1} a_k e^{jk \frac{2\pi}{N} n}\right) \\ &= \sum_{k=0}^{N-1} a_k \mathcal{F}\left(e^{jk \frac{2\pi}{N} n}\right). \end{aligned}$$

So all we need to know is

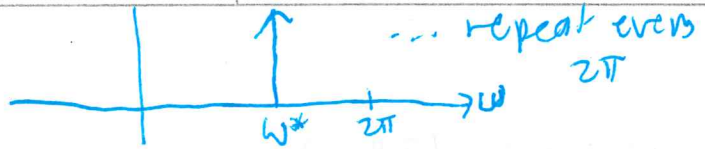
$$\mathcal{F}\left(e^{jk \frac{2\pi}{N} n}\right) = ? \quad , \quad k=0,1,2,\dots,N-1$$

Let's try with the definition of DTFT: ~~Bad! do NOT do!~~

$$\begin{aligned} &= \sum_{n=-\infty}^{\infty} e^{jk \frac{2\pi}{N} n - j\omega n} \\ &= \sum_{n=-\infty}^{\infty} e^{j(k \frac{2\pi}{N} - \omega)n} = \sum_{n=-\infty}^{\infty} (e^{j(k \frac{2\pi}{N} - \omega)n}) \end{aligned}$$

~~diverges~~

Let  $\omega^* = k \frac{2\pi}{N}$ ,



observe  $\omega^* \in [0, 2\pi)$  since  $k = 0, 1, 2, \dots, N-1$

guess  $2\pi \neq c$

$$\chi(\omega) = \begin{cases} 2\pi \delta(\omega - \omega^*), & \text{for } \omega \in [0, 2\pi) \\ \text{repeat periodically, for other } \omega. \\ \text{with period } \underline{\underline{2\pi}} \end{cases}$$

Check

$$\begin{aligned} \mathcal{F}^{-1}(\chi(\omega)) &= \frac{1}{2\pi} \int_0^{2\pi} \chi(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} 2\pi \delta(\omega - \omega^*) e^{j\omega n} d\omega \\ &= e^{j\omega^* n} \quad (\text{by sifting property of } \delta(t)) \\ &= \end{aligned}$$

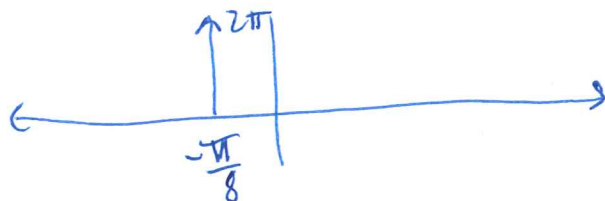
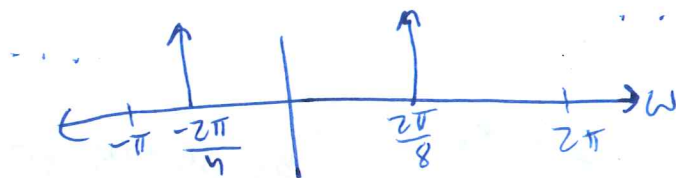
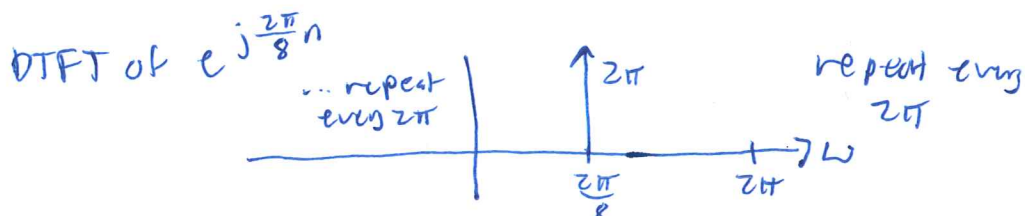
so  $e^{j\omega^* n} \xrightarrow{\text{CTFT}} \dots + 2\pi \delta(\omega - \omega^* - 2\pi) + 2\pi \delta(\omega - \omega^*) + 2\pi \delta(\omega - \omega^* + 2\pi) + 2\pi \delta(\omega - \omega^* + 4\pi) + \dots$

$$= \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega^* + 2\pi k)$$

$$X(n) = \sin\left(\frac{2\pi}{8}n\right)$$

$N=8$

$$= \frac{e^{j\frac{2\pi}{8}n} - e^{-j\frac{2\pi}{8}n}}{2j} = \frac{1}{2j} e^{j\frac{2\pi}{8}n} - \frac{1}{2j} e^{-j\frac{2\pi}{8}n}$$



$$= 2\pi \cdot \frac{1}{2j} = \frac{\pi}{j}$$

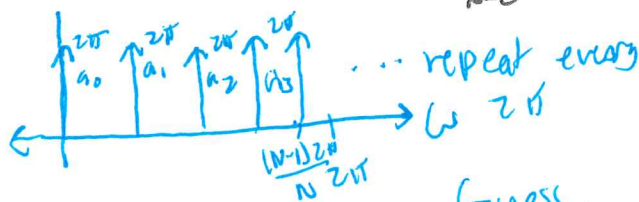
$$\sin\left(\frac{2\pi}{8}n\right) \xrightarrow{\text{DTFT}} \frac{\pi}{j} \delta\left(\omega - \frac{2\pi}{8}\right) - \frac{\pi}{j} \delta\left(\omega + \frac{2\pi}{8}\right)$$

repeat every  $2\pi$

$$= \sum_{k=-\infty}^{\infty} \left( \frac{\pi}{j} \delta\left(\omega - \frac{2\pi}{8} + 2\pi k\right) - \frac{\pi}{j} \delta\left(\omega + \frac{2\pi}{8} + 2\pi k\right) \right)$$

MUST repeat every  $2\pi$

Therefore  $\mathcal{F}\left(\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}\right) = \sum_{k=0}^{N-1} a_k \mathcal{F}\left(e^{jk\frac{2\pi}{N}n}\right)$



$$= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - k\frac{2\pi}{N} + 2\pi l)$$

Guess  
 $X(\omega) = \sum_{k=0}^{N-1} a_k 2\pi \delta(\omega - k\frac{2\pi}{N}), \text{ for } \omega \in [0, 2\pi)$

repeat w/ period  $2\pi$  for other  $\omega$ 's

$$= \sum_{k=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\frac{2\pi}{N})$$

$$\text{or } a_{k+N} = a_k$$

Obs

$$\begin{aligned} \mathcal{F}^{-1}\left(\sum_{k=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\frac{2\pi}{N})\right) &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{N-1} a_k 2\pi \delta(\omega - k\frac{2\pi}{N}) e^{j\omega n} d\omega \\ &= \sum_{k=0}^{N-1} a_k \underbrace{\int_0^{2\pi} \delta(\omega - k\frac{2\pi}{N}) e^{j\omega n} d\omega}_{= e^{jk\frac{2\pi}{N}n}, \text{ by sifting prop.}} \\ &= \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \\ &\quad \underline{\hspace{10em}} \\ &\quad X(n) \end{aligned}$$

$$\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \xrightarrow{\text{DTFT}} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\frac{2\pi}{N})$$

#### 4. Frequency Response of DT LTI systems

Recall: for LTI systems

$$e^{j\omega n} \rightarrow \boxed{h[n]} \rightarrow e^{j\omega n} H(\omega)$$

$$\text{where } H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$H(\omega)$  is the DTFT  
of  $h[n]$

Now we know that, more generally

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega \rightarrow \boxed{h[n]} \rightarrow y[n] = \frac{1}{2\pi} \int_0^{2\pi} Y(\omega) e^{j\omega n} d\omega$$

or, viewing system in frequency domain

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) = X(\omega) H(\omega)$$

CT

$$e^{j\omega t} \rightarrow \boxed{\phantom{}} \rightarrow e^{j\omega t} H(\omega)$$

$$e^{st} \rightarrow \boxed{\phantom{}} \rightarrow e^{st} H(s)$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \quad \text{freq resp}$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \quad \text{T.F.}$$

$$* H(j\omega) = H(\omega)$$

$$X(\omega) = \underline{\underline{X(e^{j\omega})}}$$

$X(s)$  Laplace Trans.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} \quad \text{freq resp.}$$

$$e^{j\omega n} \rightarrow \boxed{\phantom{}} \rightarrow e^{j\omega n} H(\omega) \quad \text{freq resp.}$$

$$z^n \rightarrow \boxed{\phantom{}} \rightarrow z^n H(z) \quad \text{transfer func.}$$

DT

$$H(\omega) = H(e^{j\omega})$$

# Causal

One can define a causal LTI system using a constant coefficient difference equation

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

Examples:  $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$

$\mathcal{F}$  ( L.H.S. ) =  $\mathcal{F}$  (R.H.S.)

$$Y(\omega) - \frac{3}{4}\mathcal{F}(y[n-1]) + \frac{1}{8}\mathcal{F}(y[n-2]) = 2X(\omega)$$

\* recall  $\mathcal{F}(x[n-n_0]) = e^{-j\omega n_0} X(\omega)$

$$\Rightarrow Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

$$Y(\omega)(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}) = 2X(\omega)$$

$$Y(\omega) = \underbrace{\frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}}_{H(\omega)} X(\omega)$$



(-) sign

$$\mathcal{F}(X(n-1)) = \sum_{n=-\infty}^{\infty} x(n-1) e^{-j\omega n}$$

let  $n = m-1$

$$\sum_{m=-\infty}^{\infty} x(m) e^{-j\omega(m+1)}$$

$$= e^{-j\omega} X(\omega)$$

\* TEST !!!

P.F. "hardest part"

- skeleton of solution

Trick to obtain the frequency response of DT LTI systems defined by difference equations.

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

$$\Rightarrow \mathcal{F} \left( \sum_{k=0}^N a_k y[n-k] \right) = \mathcal{F} \left( \sum_{m=0}^M b_m x[n-m] \right)$$

$$\Rightarrow \sum_{k=0}^N a_k \mathcal{F} ( y[n-k] ) = \sum_{m=0}^M b_m \mathcal{F} ( x[n-m] )$$

$$\Rightarrow \sum_{k=0}^N a_k e^{-j\omega k} \underline{Y(\omega)} = \sum_{m=0}^M b_m e^{-j\omega m} \underline{X(\omega)}$$

$$\Rightarrow \underline{Y(\omega)} = \frac{\sum_{m=0}^M b_m e^{-j\omega m}}{\sum_{k=0}^N a_k e^{j\omega k}} \underline{X(\omega)}$$

$H(\omega)$

So

$$H(\omega) = \frac{\sum_{m=0}^M b_m e^{-j\omega m}}{\sum_{k=0}^N a_k e^{j\omega k}}$$

Find unit impulse response?

write  $H(\omega) \rightarrow$  P.F.D partial fractions

$$H(\omega) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

hint:  $a^n u(n) \xrightarrow{\text{DTFT}} \frac{1}{1 - ae^{-j\omega}}$

$$= \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

scratch paper

$$\frac{2}{1 - \frac{3}{4}x + \frac{1}{8}x^2} = \frac{2}{(1 - \frac{1}{2}x)(1 - \frac{1}{4}x)}$$

$$= \frac{A}{1 - \frac{1}{2}x} + \frac{B}{1 - \frac{1}{4}x}$$

P.F.D  $\rightarrow$   $A=4$   
 $B=-2$

\* if run out of points on test

$\rightarrow$  just leave  $A$  &  $B$  as variables

(only lose few pts)

$$= \frac{A}{1 - \frac{1}{2}x} + \frac{B}{1 - \frac{1}{4}x}$$

$$h[n] = A\left(\frac{1}{2}\right)^n u[n] + B\left(\frac{1}{4}\right)^n u[n]$$

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] + (-2)\left(\frac{1}{4}\right)^n u[n]$$

$$X[n] = a^n u[n]$$

$$X(\omega) = \sum_{n=0}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \begin{cases} \frac{1}{1 - ae^{-j\omega}} & \text{if } |ae^{-j\omega}| < 1 \\ \text{else} & \end{cases}$$

$$\text{but } |ae^{-j\omega}| = |a| |e^{-j\omega}| \\ = |a| \cdot 1 \\ = |a|$$