LAB #3

The Existence and Uniqueness Theorems

Goal: Determine under what circumstances solution curves for a differential equation $x' = f(t,x)$ can cross; examine when can a solution not exist and when there are multiple solutions; use Existence and Uniqueness Theorems to derive qualitative information about solutions.

Required tools: ***dfield*** ; **Existence and Uniqueness Theorems** (Theorem 2.4.2 in §2.4 and 2.8.1 in §2.8 of the text); separable differential equations.

Discussion

The following constitute the **Existence and Uniqueness Theorems** from the text:

> Existence Theorem: *If $f(t,x)$ is defined and continuous on a rectangle R in the tx-plane, then given any point $(t_0, x_0) \in R$, the initial value problem*
>
> $$x' = f(t,x) \quad \text{and} \quad x(t_0) = x_0$$
>
> *has a solution $x(t)$ defined in an interval containing t_0.*

> Uniqueness Theorem: *If $f(t,x)$ and $\dfrac{\partial f}{\partial x}$ are both continuous on a rectangle R in the tx-plane, $(t_0, x_0) \in R$, and if both $x(t)$ and $y(t)$ satisfy the same initial value problem*
>
> $$x' = f(t,x) \quad \text{and} \quad x(t_0) = x_0\,,$$
>
> *then as long as $(t, x(t))$ and $(t, y(t))$ stay in R, we have*
>
> $$x(t) = y(t)\,.$$

(The **Uniqueness Theorem** asserts that if $f(t,x)$ and $\dfrac{\partial f}{\partial x}$ are continuous on a rectangle R, then solutions to the differential equation $x' = f(t,x)$ cannot cross in R.)

Assignment

(1) Choose a differential equation $x' = f(t,x))$ where $f(t,x)$ is continuously differentiable for all t and x. (Make it explicit! For example you might choose $x' = tx^2$, but you should choose something different.) Use ***dfield*** to plot a few solutions to the equation. You should find that none of your solution curves cross each other. Explain how this is a consequence of the **Existence and Uniqueness Theorems**.

(2) You will investigate the behavior of the differential equation $tx' = 2x$ in this part.

- **(a)** Use ***dfield*** to plot the direction fields and some solutions for the differential equation $tx' = 2x$. What kind of curves do they seem to be? Where do the solutions seem to cross? Turn in your graphs.
- **(b)** Find the general solution analytically to $tx' = 2x$ and show that the curves really do cross where your graph indicates. (This is a separable differential equation.) Which of the hypotheses of the **Existence and Uniqueness Theorems** does this equation <u>not</u> satisfy at the crossing point?
- **(c)** Your supervisor tells you to find the solution to the differential equation $tx' = 2x$ such that $x(1) =$ **seed**. What is your answer? (Use the solution found in Part (b) above.)
- **(d)** Your supervisor tells you to find the solution to $tx' = 2x$ which satisfies $x(0) = 0$. What is your answer? Relate your answer to the **Existence and Uniqueness Theorems**.
- **(e)** Your supervisor now wants you to find the solution to $tx' = 2x$ such that $x(0) =$ **seed**. What is your answer? (Use the solution found in Part (b) above.) Relate your answer to the **Existence and Uniqueness Theorems**.

(3) Your supervisor tells you to find the solution to the initial value problem

$$\frac{dx}{dt} = x^2\,,\ \ x(0) = 1\,.$$

Explain why you expect the solution to exist and be unique. You tell this to your supervisor and she says, "Great! What is $x(2)$?" Use ***dfield*** to attempt to answer this question. Try using larger and larger intervals for x in ***dfield*** . Why can you still not give her an answer ? Do not turn in your graphs. Rather, state the windows (range of values for t and x) you used and what you observed. At approximately which value of t does your solution leave the viewing window ? This exercise demonstrates an important limitation on the **Existence and Uniqueness Theorems**. They only guarantee existence for values of the independent variable near the initial value. Even for very nice equations and very nice initial values, the solution might not exist beyond certain limits.

(4) Solve the initial value problem in Part (3). You should get $x(t) = \dfrac{1}{1-t}$. From this it seems to say that you *can* answer her question: $x(2) = \frac{1}{1-2} = -1$. This answer is <u>wrong</u>. To understand why, use ***dfield*** (with an appropriate window) to plot the solution to the initial condition $x(0) = 1$ (call this Graph #1) and the solution to $x(2) = -\frac{1}{\{\textbf{Your Age}\}}$ (using 3 decimal places for $x(2)$ and call this Graph #2). Get both printed. Let $x(t)$ be the piecewise defined function whose graph is given by Graph #1 for $t < 1$ and by Graph #2 for $t > 1$. Trace (by hand) the graph of $x(t)$ on Graph #1. (Use a different color.) What is the formula for this function for $t < 1$? For $t > 1$? Explain why this function also satisfies the initial value problem $x' = x^2$, $x(0) = 1$ for all t (except $t = 1$). Hence $x(2) = -\frac{1}{\{\textbf{Your Age}\}}$ is just as good of an answer to your supervisor as $x(2) = -1$! Turn in your graph along with your work.

(5) Verify that the constant functions $x(t) = 0$, $x(t) = 2$ and $x(t) = 4$ are solutions to the differential equation

$$\frac{dx}{dt} = \frac{te^{-x}(x-4)(2x-x^2)}{x-3} .$$

Now consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{te^{-x}(x-4)(2x-x^2)}{x-3} \\ \\ x(3) = \frac{3}{2} \end{cases}$$

Using the **Uniqueness Theorem**, can you conclude that the solution $x(t)$ is bounded (i.e., $\alpha < x(t) < \beta$) ? If so, what are the values of α and β ? Use ***dfield*** to plot the solution of the above initial value problem. Does this support your claim ?

(Note that even though the **Existence Theorem** guarantees the above initial value problem has a solution, we cannot solve for $x(t)$ explicitly. However, we can still obtain useful qualitative information about its solution using the **Uniqueness Theorem**.)