Causality of LTI systems

Tuesday, September 11, 2007
3:11 PM

<u>Fact</u> When an LTI System is causal, then

(DT) $h[n]=0$, for $n<0$
(CT) $h(t)=0$, for $t<0$

why? $y[n]=\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$ so this must be zero when $k>n$

i.e. $h[\text{neg int}]=0$

y[n] does not depend on x[k] for k>n

corollary - output of causal linear, time-invariant system is

$$y[n]=\sum_{k=-\infty}^{n} x[k]h[n-k]$$

stability:

fact when the system is LTI, it is stable iff. $\sum_{k=-\infty}^{\infty}|h[k]|$ is finite

$\int_{-\infty}^{\infty}|h(\tau)|d\tau$ is finite

why? assume x(t) is bounded i.e. $|x[n]|<\varepsilon$

$$|y[n]|=\left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| = \left|\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$=\sum |h[k]|\,|x[n-k]| \le$$

is $<\varepsilon$

so $<\sum_{-\infty}^{\infty}|h[k]|\,\varepsilon = \varepsilon\sum_{-\infty}^{\infty}|h[k]|$

so just make sure this is bounded

ε is bounded by definition

Q: An LTI system has $h[n]=\delta[n]$
show that the system is stable

so $\sum_{-\infty}^{\infty}\delta[n]=1$ so stable

Q Show that if $y(t)=\int_{-\infty}^{\infty}x(\tau)\,d\tau$ then system is unstable

Hint: observe that $h(t) = u(t)$

The system is LTI so all we need to check is if $\int_{-\infty}^{\infty} h(\tau)\,d\tau$ is finite.

To get $h(t)$, plug $\delta(t)$ into output formula:

$$y(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} |u(t)|\,dt = \int_{-\infty}^{\infty} u(t)\,dt = \infty$$, so it is unstable

show LTI: 1) To show linearity:

$$a x_1(t) + b x_2(t) \longrightarrow \int_{-\infty}^{t} a x_1(\tau) + b x_2(\tau)$$

$$= a\int_{-\infty}^{t} x_1(\tau)\,d\tau + b\int_{-\infty}^{t} x_2(\tau)\,d\tau$$

$$= a y_1(t) + b y_2(t)$$

2) Time Invariance:

$$x(t) \to \boxed{TD} \to z(t) = x(t-t_0) \to \boxed{sys} \to \int_{-\infty}^{t} z(\tau)\,d\tau = \int_{-\infty}^{t} x(\tau - t_0)\,d\tau$$

$$x(t) \to \boxed{sys} \to y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \to \boxed{TD} \to z(t) = y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\,d\tau$$

let $\theta = \tau - t_0$ then $d\theta = d\tau \Rightarrow \int_{-\infty}^{t-t_0} x(\theta)\,d\theta$

which is same as $\int_{-\infty}^{t-t_0} x(\tau)\,d\tau$