

12 JANUARY 2012

MA 375

Wrong Use of Induction

"Thm": "All horses have the same color."

Proof: Let $P(n)$ = "Within any group of n -horses, all horses are of the same color."

(1) Base Case:

$n=0$: nothing to prove ($n=0 = 0$ horse: no coloring...)

$n=1$: claim is evident (group of 1 horse does have all the same color, namely its own)

(2) "Crank" (Inductive Step): Assume that the claim is true for all group of n horses, and prove it for all groups of $n+1$ horses

Let S be a group of $n+1$ horses, namely

$H_1, H_2, \dots, H_n, H_{n+1}$

Consider 2 ^{sub} groups

$T_1 = H_1, \dots, H_n$

$T_2 = H_2, \dots, H_{n+1}$

\sqsubset both group have size n

By inductive hypothesis all horses in T_1 and T_2 have the same color C_1 . Similarly, all horses in T_2 have the same color C_2 .

However, H_2, \dots, H_n are in both groups

$\Rightarrow C_1 = C_2 \quad \square$

\hookrightarrow All Horses Have The Same Color?

Problem: If $n=1$, there is no overlap as assumed here.

Thus inductive proof does not work.

* ~~not~~ On "horses are not well ordered"

= not really; we are concerned w/ $n \in \mathbb{N}$ anyhow which is well ordered.

* Do not apply result of inductive proof for $n \notin \mathbb{N}_{\text{not}}$

Q: Prove that for $n \in \mathbb{N}$

$$4 \mid (7^n - 3^n)$$

Proof: Base case: $n=0$, $4 \mid 7^0 - 3^0 = 0 \checkmark$

$$n=1, \quad 4 \mid 7^1 - 3^1 = 4 \checkmark$$

$$n=2, \quad 4 \mid 7^2 - 3^2 = 40 \checkmark$$

Inductive step: Assume that $4 \mid 7^i - 3^i$ is true for

$$i = 0, 1, \dots, n \text{ and deduce that } 4 \mid 7^{n+1} - 3^{n+1}$$

→ Strategy: construct a meaningful equation from above statement

$$\textcircled{1} 4 \mid 7^n - 3^n \text{ means } 4 \cdot k_n = 7^n - 3^n \text{ where } k_n \in \mathbb{N}$$

Goal $4 \cdot k_{n+1} = \textcircled{2} 7^{n+1} - 3^{n+1}$

How can write $\textcircled{2}$ from $\textcircled{1}$?

Experiment

(1) Multiply each side by $7+3$

$$(4 \cdot k_n = 7^n - 3^n) \cdot 7+3$$

IS THIS DIV. BY 4?

$$= 4 \cdot k_n (7+3) = (7^{n+1} - 3^{n+1}) + 3 \cdot 7^n - 7 \cdot 3^n$$

→ What is divisible by 4?

$$(1) 4 \cdot n$$

$$(2) 7^n - 3^n \text{ (we assumed this) or } m(7^n - 3^n), m \in \mathbb{N}$$

so

$$\begin{aligned} (7^{n+1} - 3^{n+1}) &= 4 \cdot k_n (7+3) + 3 \cdot 7^n - 7 \cdot 3^n \\ &= 4 \cdot k_n (7+3) \end{aligned}$$

$$\begin{aligned} (7+3)(7^n - 3^n) &= (7^{n+1} - 3^{n+1}) + 3 \cdot 7^n - 7 \cdot 3^n \\ &= (7^{n+1} - 3^{n+1}) + \boxed{3 \cdot 7^n - 7 \cdot 3^n} - \boxed{4 \cdot 3^n} \end{aligned}$$

Therefore

$$7^{n+1} - 3^{n+1} = 7(7^n - 3^n) + 4 \cdot 3^n \quad \square$$

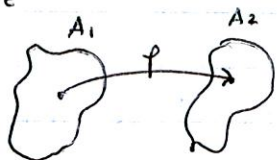
(we make last assumption that sum of two number divisible by n is also divisible by n)

Inclusion / Exclusion

Setup: A_1 and A_2 are sets

→ on the size of sets

suppose



function: from each element of A_1 emanates exactly one arrow terminating at some element of A_2 .

special considerations

- (1) If all elements of A_2 are the end of at least one arrow, f is called "surjective" or "onto"
- (2) If no elements of A_2 is the end of two or more arrows, f is called "injective" or "into"
- (3) If f is both "into" and "onto", it is "bijective"
= elements on $A_1 \leftrightarrow A_2$ form a 1 to 1 correspondence

examples

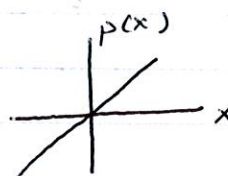
(1) $A_1 = \mathbb{R}$

$p(x) = x$

$A_2 = \mathbb{R}$

→ assignment of x in A_1 to A_2

→ function is bijective

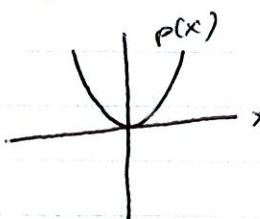


(2)

$p(x) = x^2$

= from each $x \in A_1$ emanate "only 1 arrow"

= Some A_2 have two arrows. Some have none

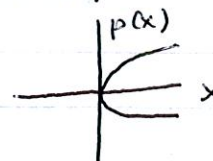


(3) p

$p(x) = \sqrt{x}$

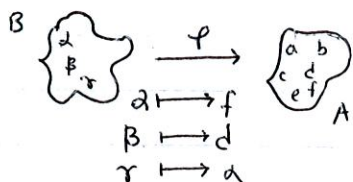
= not even a function

= no output for $x < 0 \mid x \in A$



We want to compare size of the sets

Def. We write $A \geq B$ if there's a function $f: B \rightarrow A$ that is injective



(recall: injective = no two arrows collide)

Def A and B are equal size if $A \geq B$ and $B \geq A$.

ex $A = \mathbb{Z}$ $B = \mathbb{Q}$

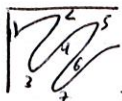
= intuition tells us that there are more elements in \mathbb{Q} than \mathbb{Z}
= however...

note that we construct all possible \mathbb{Q} in following manner

$q \backslash p$	1	2	3	...
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	
...				

→ every positive rational numbers show at least once in the table

→ we will count & number all table entry
i.e.



= We have labeled all rational #s with integer
thus $\mathbb{Q} \leq \mathbb{Z}$ thus $|\mathbb{Q}| = |\mathbb{Z}|$

Ex $\mathbb{Q} = A$ $\mathbb{R} = B$ = Note now that \mathbb{R} is larger than \mathbb{Q}
surely, $\mathbb{Q} \leq \mathbb{R}$

= ∞ of \mathbb{Q} is bigger than the ∞ of \mathbb{R} ? (diagonal method of Cantor continuum hypothesis)

↳ follow-up Q: is there a set whose size is between \mathbb{Q} & \mathbb{R} ?

⇒ cannot be answered. Diff mathematical const. can be built for both "yes" and "no"

Recall A_1, A_2 are sets

Def $A_1 \cap A_2$ = All elements A_1 that also live in A_2 = intersection
 $A_1 \cup A_2$ = All elements in A_1 or A_2 or both = union

We denote size of A by notation $|A|$

Then, what is the size of $|A_1 \cup A_2|$?

= $|A_1| + |A_2|$? Not quite.

⇒ if somebody lived in $A_1 \cap A_2$, he gets counted twice
= $|A_1| + |A_2| - |A_1 \cap A_2|$

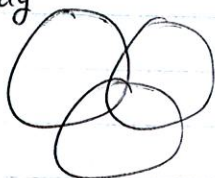
elements	counted in			
	A_1	A_2	$A_1 \cap A_2$	
$A_1 \setminus A_2$ ($A_1 \text{ not } A_2$)	1	0	0	= 1
$A_2 \setminus A_1$	0	1	0	= 1
$A_2 \cap A_1$	1	1	-1	= 1

} we want all 3 values to be 1

For 3 sets

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Venn Diag



= check if everyone is counted once.