

# LAB #6

## THE SWAYING BUILDING

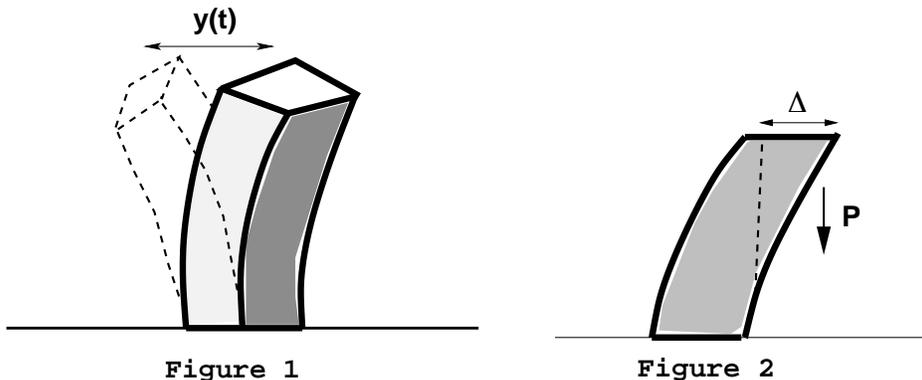
**Goal:** Determine a model of the swaying of a skyscraper; estimating parameters

**Required tools:** MATLAB routines *pplane*, *ode45*, *plot*; M-files; systems of differential equations.

### DISCUSSION

Modern skyscrapers are built to be flexible. In strong gusts of wind or in earthquakes these buildings tend to sway back and forth to absorb the shocks. Oscillations with an amplitude on the order of 5 to 10 seconds are common. You will analyse two different differential equations that model the swaying of a building.

Let  $y(t)$  be a measure of how far the building is bent- the displacement (in meters) of the top of the building with  $y = 0$  corresponding to the perfectly vertical position. When  $y \neq 0$ , the building is bent and the structure applies a strong restoring force back toward the vertical (see Figure 1).



This is reminiscent of a harmonic oscillator and therefore a very crude approximation of the motion of a swaying building is the damped harmonic oscillator equation:

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

Here the constants  $p$  and  $q$  are chosen to reflect the characteristics of the particular building being studied.

**The P-Delta Effect.** Modeling the swaying building with a harmonic oscillator equation is extremely crude. We do not claim that the forces present in a swaying building are identical to those of a spring. The harmonic oscillator is only a first approximation of a complicated physical system. To extend the usefulness of the model, we must consider other factors that govern the motion of the swaying building. One aspect of the model of the swaying building that we have not yet included is the effect due to gravity. When the building undergoes small oscillations, gravity does not play a very important role. However, if the oscillations become large enough, then gravity can

have a significant effect. When  $y(t)$  is at its maximum value, a portion of the building is not directly above any other part of the building (see Figure 2).

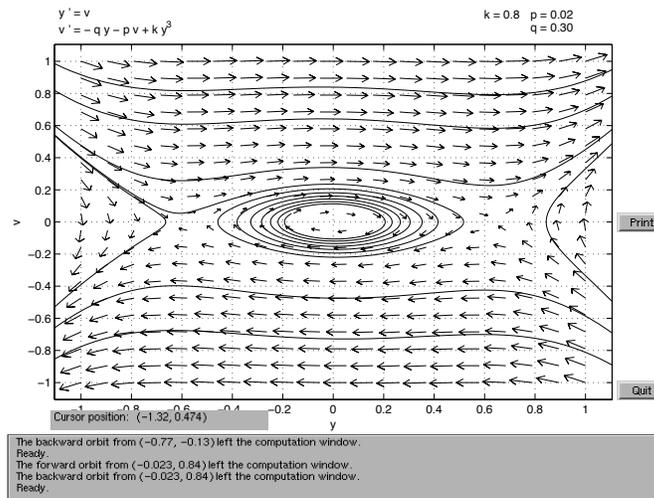
Therefore, gravity pulls downward on this portion of the building and this force tends to bend the building farther. This is called the “P-Delta” effect ( $\Delta$  is the overhang distance and  $P$  is the force of gravity). To include this effect in our model in a way that is quantitatively accurate requires knowledge of the density of the building and the flexibility of the construction materials. Without going into specific details, we can construct a much simplified model that is a caricature of the P-Delta effect.

The P-Delta effect is very small, much smaller than the restoring force. As  $y$  increases, the P-Delta effect becomes quite large. As a first model, we may assume the force provided by the P-Delta effect is proportional to  $y^3$ . Adding this force corresponds to adding a term to the expression for the acceleration of  $y$ , that is, adding a term proportional to  $y^3$  to the right-hand side of the 2<sup>nd</sup> order differential equation. For the system, this corresponds to adding a term  $ky^3$  to the equation :

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = ky^3 .$$

Let  $y_1(t)$  and  $y_2(t)$  represent measurements of displacement as a function of time of the building reacting to two different shocks (Shock #1, Shock #2):

$t$	$y_1(t)$	$y_2(t)$
0	0.14	0.50
2	0.38	0.40
4	0.21	-0.12
6	-0.18	-0.49
8	-0.37	-0.33
10	-0.15	-0.33
12	0.21	0.48
14	0.34	0.25
16	0.10	-0.23
18	-0.24	-0.45
20	-0.31	-0.18
22	-0.05	0.27



## ASSIGNMENT

You will basically do the following in this lab:

- (i) Determine which equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \quad (\text{A})$$

or

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = ky^3 \quad (\text{B})$$

forms a better model for the given data.

- (ii) Approximate values of the parameters  $p, q$ , and  $k$  so that the solutions to (A) and/or (B) fit the observed data reasonably well.
- (iii) With your choice of parameters, comment on how well the models agree with the data above.
- (iv) Finally, determine if there is any danger the building will collapse in a severe hurricane or earthquake. How can you tell from your graphs whether the building is going to collapse ?

The physical meaning of the constants  $p, q$ , and  $k$  are as follows:  $p$  is a measure of damping (essentially controls the magnitude of  $y$ );  $q$  is a measure of stiffness (it controls the rapidity of vibrations);  $k$  measures the effect due to gravity (essentially controls the fine tuning of the magnitude). These interpretations will be useful in trying to estimate these parameters.

The basic technique is to “guess” values of the constants  $p, q$  and  $k$ , in a systematic way, and then use these constants to plot the solution to the differential equations (A) or (B) and the actual data on the same graph. You then change the parameters, plot the solution to the differential equation (with the updated choices of parameter) and the actual data on the same graph. Continue adjusting parameters until the computed solution agrees reasonably well with the actual data.

Here is a systematic approach. Observe that from the actual data above, the displacements  $y_1(t)$  due to Shock # 1 tend to be smaller than displacements  $y_2(t)$  due to Shock #2. We expect that the effect of gravity is minimal for small displacements. Hence  $k = 0$  and we proceed as follows:

Step 1: Set  $k = 0$  and  $p = 0$ . Choose a random value of  $q$ , where  $0 \leq q \leq 1$  (Recall that  $v_0 = \frac{y_1(2) - y_1(0)}{2} = 0.12$  is an estimate for the initial velocity and  $y_0 = 0.14$  is the true initial position.) Plot the solution of (A) and the true data  $y_1$ .

Step 2: Adjust  $q$  (stiffness) until the period of the solution gives a reasonable approximation to the period of the  $y_1$  data.

Step 3: Adjust the initial velocity  $v_0$  until the solution looks reasonably close to the  $y_1$  data.

Step 4: Adjust  $p$  (damping) until the amplitude of the solution is reasonably close to the  $y_1$  data. (Let  $0 \leq p \leq 1$ )

Step 5: Record your final values of  $q$ ,  $v_0$ , and  $p$  (Remember,  $k = 0$ ).

Step 6: Using the *same values* of  $p$  and  $q$  from Step 5, compare the solution of (A) to the  $y_2$  data. Of course, in this case here  $y_0 = 0.50$  and an approximation to  $v_0 = \frac{y_2(2) - y_2(0)}{2} = -0.05$  and you may need to adjust  $v_0$ , as in Step 3, to get a reasonable approximation. If the approximation is bad, try increasing  $k$  ( $0 \leq k \leq 1$ ) until a good approximation is obtained. (Hence when  $k \neq 0$ , you are now solving equation (B).)

For your graphs, you will first use the MATLAB program **ode45** to solve the differential equation (B). To use **ode45**, you will need to create a function file which represents the differential equation as a system. If we let  $y = y(t)$  and  $v = y'(t)$ , then equation (B) is equivalent to the system

$$\begin{cases} y' = v \\ v' = -qy - pv + ky^3 \end{cases} .$$

As a matrix equation, this becomes

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ ky^3 \end{bmatrix} .$$

For MATLAB to deal with this equation, you must first choose values for the constants. Since you will be adjusting these constants many times, there is a technique for doing this efficiently. First create an *m-file* as follows. In MATLAB pull down the “File” menu and select “New M-File”. In the window that will appear, enter the following statements:

```
function g=yy(t,x)
global k;
global p;
global q;
A=[0, 1; -q, -p];
g=A*x+[0; k*x(1)^3];
```

You then select “Save As” and save it as “**yy.m**”.

Remark: The “global” command will allow this m-file to share the values of  $k, p, q$  with your main program. It will save time and typing.

For your plotting, first create a vector  $\mathbf{s}$  which represents the time variable. A convenient way of doing this is with the command:

```
>> s=0:2:22;
```

Next you create vectors  $y_1$  and  $y_2$  which represent the given data:

```
>> y1=[0.14,0.38 ,..., -0.05];
```

```
>> y2=[0.50,0.40 ,..., 0.27];
```

The initial position is  $y_0 = 0.14$  and the average velocity over the first interval is  $v_0 = \frac{0.38 - 0.14}{2} = 0.12$

Now you can begin Step (1) above. You will be changing the parameters  $k, p, q$ , and  $v_0$ . You can perturb these parameters and see the plots of the solution to equation (B) compared with the actual data  $y_1$  as follows:

```
>> global k;
>> global p;
>> global q;
>> cla;hold on;plot(s,y1,'r');k=0.0;p=0.0;q=0.2;
[t,y]=ode45('yy',[0,22],[0.14,0.12]);plot(t,y(:,1));
```

The line above should be typed as one long line, starting with `cla;` and ending with `plot(t,y(:,1));`. The red graph is the graph of the  $y_1$  data points, the blue graph is the approximation. Now to change parameters and view updated plots, simply use the  $\uparrow$  key. The long line above should appear. Simply change whichever parameters you desire and hit “Enter” or “Return”. Continue this process of adjusting parameters using the  $\uparrow$  key until you are satisfied that your values of  $p, q$  will make the solution to the differential equation fit the  $y_1$  data reasonably well. (Remember you may need to adjust and  $v_0$ .)

With the *same* values of  $p, q$  you found to fit the  $y_1$  data, now perturb  $k$  until the solution to (B) fits the  $y_2$  data reasonably well.

These will allow you to answer (i)-(iii). To address (iv), you might want to enter the system (with the values of  $p, q$  and  $k$  you found) into ***pplane*** which will compute the phase plane portrait of the solution which, in this case, is a plot of velocity against position, and compute a few of the orbits to see if you can make the building collapse with the right initial conditions. Regarding the question, “How can you tell from your graphs whether the building is going to collapse?” look at the phase portrait on Page 2 of this lab. What is happening to the displacement if the initial data of the building is described by a point on one of the spiral orbits? What if the initial data is described by a point on one of the other orbits?