(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{2j}\right)^{|n|}.$$

$\frac{1}{2j}^{n} \quad n > 0 \qquad \frac{1}{2j}^{-n} \quad n < 0$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2j}^{n} u[n]\right) e^{-j\omega n} + \left(\frac{1}{2j}^{-n} u[-n]\, e^{-j\omega n}\right) \right]$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{n} u[n]\, e^{-j\omega n} + \sum_{n=-\infty} \left(\frac{1}{2j}\right)^{-n} u[-n]\, e^{-j\omega n}$$

$u[n] = 0 \quad n < 0 \qquad u[-n] = 0 \quad n > 0$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2j}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{e^{j\omega}}{2j}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2j e^{j\omega}}\right)^{n}$$

$$= \sum_{k=1}^{\infty} \left(\frac{e^{j\omega}}{2j}\right)^{k} + \sum_{n=0}^{\infty} \left(\frac{1}{2j e^{j\omega}}\right)^{n} - \frac{1}{2j} e^{0}$$

$k = -n$

$$= \frac{1}{1 - \frac{e^{j\omega}}{2j}} - 1 + \frac{1}{1 - \frac{1}{2j e^{j\omega}}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n} + \sum_{n=-\infty}^{0} \left(\frac{1}{2j}\right)^{-n} e^{-j\omega n}$$

$$\boxed{\begin{matrix} m = -n \\ n = -m \end{matrix}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n} + \sum_{m=\infty}^{0} \left(\frac{1}{2j}\right)^{m} e^{j\omega m}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n} + \sum_{m=0}^{\infty} \left(\frac{1}{2j}\right)^{m} e^{j\omega m}$$

$$\boxed{= \frac{1}{1 - \left(\frac{1}{2j}\right) e^{-j\omega}} + \frac{1}{1 - \left(\frac{1}{2j}\right) e^{j\omega}}}$$

2

$y(t) = e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{3}e^{t}u(-t) + \frac{1}{3}e^{-2t}u(t)$

$x(t) = e^{-|t|}$

$\mathcal{X}(\omega) = \mathcal{F}(e^{-|t|})$

$= \mathcal{F}(e^{-t}u(t) + e^{t}u(-t))$

$= \mathcal{F}(e^{-t}u(t)) + \mathcal{F}(e^{-t}u(t))$ $\omega \to -\omega$

$= \frac{1}{1+j\omega} + \frac{1}{1-j\omega}$ by 7

$y(t) = \mathcal{F}^{-1}\left(\left(\frac{1}{1+j\omega} + \frac{1}{1-j\omega}\right)\frac{1}{j\omega+2}\right)$

$= \mathcal{F}^{-1}\left(\frac{1}{1+j\omega}\frac{1}{j\omega+2} + \frac{1}{1-j\omega}\frac{1}{j\omega+2}\right)$

$= \mathcal{F}^{-1}\left(\frac{1}{1+j\omega} - \frac{1}{j\omega+2} + \frac{1/3}{1-j\omega} + \frac{\frac{1}{3}}{j\omega+2}\right)$

(20 pts) **2.** The Frequency response of a continuous-time LTI system is

$$H(j\omega) = \mathcal{H}(\omega) = \frac{1}{j\omega + 2}.$$

Use the convolution property of the Fourier transform to determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$.

$\mathcal{H}(\omega) = \frac{1}{2+j\omega}$ by $*(7)$

$h(t) = e^{-2t}u(t)$

2

$\mathcal{F}^{-1}(\mathcal{X}(\omega)\mathcal{H}(\omega))$

$=$

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$

$e^{-|t|} \to e^{-t}$ $t>0$

e^{t} $t<0$

$\Downarrow$

$e^{-|t-\tau|} \to e^{-t+\tau}$ $t>0$

$e^{t-\tau}$ $t<0$

$= \int_{-\infty}^{\infty} e^{-|t-\tau|} e^{-2\tau} u(\tau)\, d\tau$ $u(\tau)=0$ $\tau<0$

$= \int_{0}^{\infty} e^{-|t-\tau|} e^{-2\tau}\, d\tau$

?

$= \int_{0}^{\infty} e^{-t+\tau} e^{-2\tau}\, d\tau$

$= \int_{0}^{\infty} e^{-t-\tau}\, d\tau$

$= \int_{0}^{\infty} e^{-t} e^{-3\tau}\, d\tau$

$= e^{-t}\int_{0}^{\infty} e^{-\tau}\, d\tau$

$= e^{-t}\left[\frac{1}{-1}e^{-\tau}\right]_0^{\infty}$

$= e^{-t}\left[0 + \frac{1}{-1}\right]$

$\boxed{= -e^{-t}}$

5

(15 pts) **3.** True/False? The Fourier transform of a DT signal $x[n]$ is a periodic function, no matter what $x[n]$ is. (Justify your answer.)

True, the fourier transform of any DT signal : is periodic w/ period 2π. From (26) the inverse transform is only taken over a period of 2π. If FT was not periodic w/ period 2π the inverse FT Definition would not hold, esp for non periodic or periodic w/ period $\geq 2\pi$ FT's.

The FT of a ~~DT~~ DT signal is periodic w/period 2π b/c it is a linear combination of functions which are periodic w/ period 2π.

10

(10 pts) **4.** A continuous-time LTI system has frequency response

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}.$$

Derive a differential equation representing this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(j\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 4}{(j\omega+2)(j\omega+3)}$$

$$Y(\omega)(j\omega+2)(j\omega+3) = X(\omega)(j\omega+4)$$

$$(Y(\omega)(j\omega) + 2Y(\omega))(j\omega+3) = X(\omega)(j\omega) + 4X(\omega)$$

$$\underset{\text{by 16}}{3Y(\omega)(j\omega)} + \underset{\text{by 16}}{Y(\omega)(j\omega)(j\omega)} + \underset{\text{by 16}}{2Y(\omega)(j\omega)} + 6Y(\omega) = \underset{\text{by 16}}{X(\omega)(j\omega)} + 4X(\omega)$$

$$3\frac{d}{dt}Y(\omega) + \frac{d^2}{dt^2}Y(\omega) + 2\frac{d}{dt}Y(\omega) + 6Y(\omega) = \frac{d}{dt}X(\omega) + 4X(\omega)$$

$$\boxed{\frac{d^2}{dt^2}Y(\omega) + 5\frac{d}{dt}Y(\omega) + 6Y(\omega) = \frac{d}{dt}X(\omega) + 4X(\omega)}$$

2

$\mathcal{X}(\omega) = -2e^{(j-1)\omega} u(\omega+1)$

$x(t) = \frac{1}{2\pi}(-2)\int_{-\infty}^{\infty} e^{j\omega-\omega} e^{j\omega t} d\omega$

$= \frac{-1}{\pi}\int_{-\infty}^{\infty} e^{(j-1)\omega} e^{j\omega t} d\omega$

$= \frac{-1}{\pi}\left[\frac{e^{(j-1+jt)\omega}}{j-1+jt}\right]_{-1}^{\infty}$

$= \frac{-1}{\pi}\cdot\left[0 - \frac{e^{-j+1-jt}}{j-1+jt}\right]$

$\frac{+1}{\pi}\left(\frac{e^{-j+1-j(t-6)}}{j-1+j(t-6)}\right)$ * time shift

(10 pts) **5.** A CT signal $x(t)$ has Fourier transform

$$\mathcal{X}(\omega) = -2e^{(j-1)\omega}u(\omega+1).$$

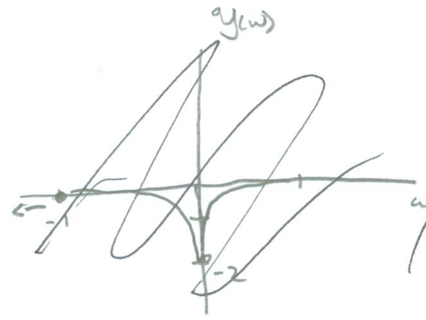
Denote by $y(t)$ the signal obtained by delaying $x(t)$ by six seconds. Sketch a graph representing the magnitude $|\mathcal{Y}(\omega)|$ of the Fourier transform $\mathcal{Y}(\omega)$ of $y(t)$. (Justify your answer.)

$\mathcal{X}(\omega) = -2e^{j\omega-\omega} u(\omega+1)$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -2$

$\mathcal{X}(\omega) = -2e^{(j-1)\omega} u(\omega+1)$ * time shift (10) *

$\mathcal{Y}(\omega) = e^{-j\omega 6}\left(-2e^{(j-1)\omega} u(\omega+1)\right)$ ✓

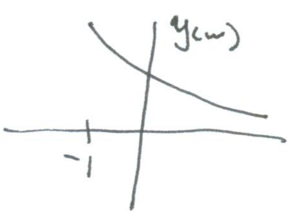


$|\mathcal{Y}(\omega)| = |\mathcal{X}(\omega)|$

$|\mathcal{X}(\omega)| = |-2e^{(j-1)\omega} u(\omega+1)|$

$= 2|e^{j\omega}e^{-\omega}| u(\omega+1)$

$= 2e^{-\omega} u(\omega+1)$



~~$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{Y}(\omega) e^{j\omega t} d\omega$~~

~~$= \frac{1}{2\pi}(-2)\int_{-\infty}^{\infty} e^{-j\omega 6} e^{(j-1)\omega} u(\omega+1) e^{j\omega t} d\omega$~~

~~$= \frac{-1}{\pi}\int_{-\infty}^{\infty} e^{-\omega} e^{-5j\omega} u(\omega+1) e^{j\omega t} d\omega$~~

~~$= \frac{-1}{\pi}\int e^{-\omega(1+5j)} e^{j\omega t} d\omega$~~ $u(\omega+1) = 0$, $\omega < -1$

~~$= \frac{-1}{\pi}\int_{-1}^{\infty} e^{+\omega(-1-5j+jt)} d\omega$~~

~~$= \frac{-1}{\pi}\left[\frac{e^{(-1-5j+jt)\omega}}{(-1-5j+jt)}\right]_{-1}^{\infty}$~~