ECE 302

Midterm Examination 1

Summer 2016

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Instructions:

1. Do not open your exam booklet until the BEGIN signal is given.
2. Enter your name, ID #, and signature on the space provided below. Your signature indicates that you have not received unauthorized aid on the exam.
3. The exam is closed book and closed notes. Calculators are allowed.
4. An equation sheet is provided on the last page of the exam booklet.
5. There are 6 multiple choice problems (10 points each). You must CLEARLY indicate your answer to receive credit. No partial credit will be given for these problems.
6. There are 2 work-out problems (20 points each). You must show work to receive any credit. Partial credit will be given at the discretion of the instructor. Clearly designate final answers.
7. You have 60 minutes to complete this exam. When the end of the exam is announced, you must stop writing immediately. Anyone caught writing after the exam is over will get a grade of zero.

Name: **SOLUTION**

ID #:

Signature:

Multiple Choice Section

Questions 1 - 6 are multiple choice problems (10 points each). You must clearly select a single answer for each problem to receive credit. No partial credit will be given for these problems.

1. Let A and B be events with $\Pr(A) = 0.7$, $\Pr(B) = 0.3$, and $\Pr(A \cap B) = 0.2$. Which of the following statements is (are) true?

   I. A and B are independent.

   II. A and B are mutually exclusive.

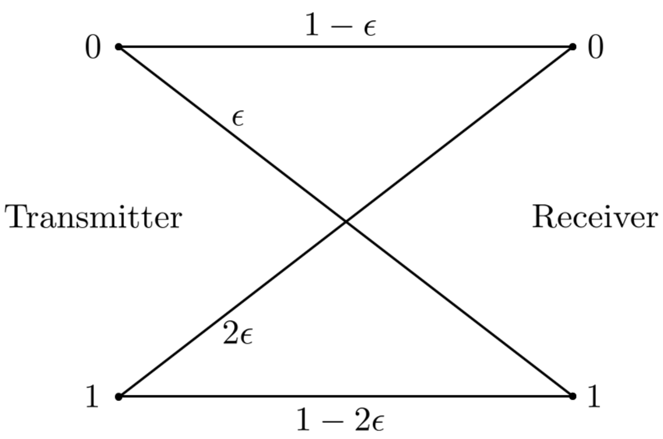
   III. A and B are collectively exhaustive.

   (A) Only I.
   (B) Only II.
   (C) Only III.
   (D) Only I and II.
   (E) Only I and III.
   (F) Only II and III.
   (G) I, II, and III.
   $\boxed{\text{(H)}}$ None of the statements are true.

2. Let A, B, C be events, where A and $B \cup C$ are mutually exclusive and collectively exhaustive and B and C are independent. Suppose that $\Pr(A) = 1/2$ and $\Pr(C) = 1/3$. What is $\Pr(B)$?

   (A) 1/2
   (B) 1/3
   $\boxed{\text{(C)}}$ 1/4
   (D) 1/5
   (E) 1/6

3. Consider the nonsymmetric binary channel shown below. Assume the probability of receiving a bit in error is 1/3 and the probability of sending a 0 is 1/3. What is the probability of receiving a 0?



(A) 5/12

(B) 4/9

(C) 8/15

(D) 2/3

(E) 4/5

4. Two balls are selected without replacement from an urn containing 4 black balls and 5 white balls. What is the probability that the first ball is white given that the second ball is white?

   (A) 1/6
   (B) 3/8
   (C) 4/9
   (D) 1/2
   (E) 5/9

5. A student rolls two fair dice. Assume the rolls are independent. What is the probability that one roll is a 1 and one roll is a 2 given that the first roll is even?

   (A) 1/2
   (B) 1/3
   (C) 1/6
   (D) 1/12
   (E) 1/18
   (F) 1/36

6. Let X be a random variable with cdf

$$F_X(x) = \begin{cases} 0 & , x < 0, \\ cx & , 0 \leq x < 1, \\ 1 & , x \geq 1, \end{cases}$$

where c is a constant. Which of the following statements gives all possible values that c can assume?

(A) $c = 0$

(B) $c = 1$

(C) $c \in \{0, 1\}$

(D) $c \in [0, 1]$

(E) $c \in [0, \infty)$

Work-out Section

Questions 7 and 8 are work-out problems (20 points each). You must show work to receive credit.

7. Let X be a random variable with pdf

$$f_X(x) = ce^{-4|x|},$$

where c is a constant. Let $Y = g(X)$ where

$$g(x) = \begin{cases} 1 & , -1 \leq x < 1, \\ 0 & , \text{else}, \end{cases}$$

(a) (5 points) Find c.

(b) (6 points) Find $\Pr(X^2 < 1)$.

(c) (2 points) What type of random variable is Y? Explain your answer.

(d) (7 points) Find $f_Y(y)$.

(e) (Bonus) Find $\mathbb{E}[Y]$.

Solution:

(a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_{-\infty}^{\infty} ce^{-4|x|}dx \\ &= 2c\int_{0}^{\infty} e^{-4x}dx \\ &= c/2 \\ \implies & \boxed{c = 2} \end{aligned}$$

(b)

$$\begin{aligned}
\Pr(X^2 < 1) &= \Pr(-1 < X < 1) \\
&= \int_{-1}^{1} f_X(x)dx \\
&= \int_{-1}^{1} 2e^{-4|x|}dx \\
&= 2\int_{0}^{1} 2e^{-4x}dx \\
&= \boxed{1 - e^{-4}}
\end{aligned}$$

(c) The function $g(x)$ only takes on the values 0 and 1, therefore, Y can only take on the values 0 and 1. Since the number of valeus that Y takes on is finite, Y must be a discrete random variable.

(d) Y is a discrete random variable which only takes on the values of 0 and 1 so we can write its pdf as

$$f_Y(y) = \Pr(Y = 0)\delta(y) + \Pr(Y = 1)\delta(y - 1).$$

The probabilities $\Pr(Y = 1)$ and $\Pr(Y = 0)$ are given by

$$\begin{aligned}
\Pr(Y = 1) &= \Pr(g(X) = 1) \\
&= \Pr(-1 \leq X < 1) \\
&= 1 - e^{-4} \\
\Pr(Y = 0) &= 1 - \Pr(Y = 1) \\
&= e^{-4}.
\end{aligned}$$

Therefore, $\boxed{f_Y(y) = e^{-4}\delta(y) + (1 - e^{-4})\delta(y - 1).}$

(e)

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[g(X)] \\
&= \int_{-\infty}^{\infty} g(x) f_X(x)dx \\
&= \int_{-1}^{1} f_X(x)dx \\
&= \Pr(-1 < X < 1) \\
&= \boxed{1 - e^{-}4}
\end{aligned}$$

8. A coin with probability p of coming up heads is flipped continually until both heads and tails have appeared. Assume all coin flips are independent.

(a) (12 points) Find the probability that the coin was flipped k times (specify the possible values of k).

(b) (8 points) Find the probability that the 1^{st} flip was heads given the coin was flipped k times.

Solution:

(a) Let A be the event that the coin is flipped k times. Let H_i, T_i denote that the ith flip comes up heads or tails, respectively. There are two possible outcomes that can occur when the coin is flipped k times:

- The coin comes up heads on the first $k-1$ flips and then comes up tails on the k^{th} flip - $\{H_1H_2...H_{k-1}T_k\}$
- The coin comes up tails on the first $k-1$ flips and then comes up heads on the k^{th} flip - $\{T_1T_2...T_{k-1}H_k\}$

Therefore,

$$\begin{aligned}\Pr(A) &= \Pr(\{H_1H_2...H_{k-1}T_k\}) + \Pr(\{T_1T_2...T_{k-1}H_k\}) \\ &= \boxed{p^{k-1}(1-p) + (1-p)^{k-1}p} \quad \text{, by the geometric probability law.}\end{aligned}$$

Since at least two flips are required to observe both heads and tails, $\boxed{k \geq 2}$.

(b) Let B be the event that the 1^{st} flip is heads. Want to find $\Pr(B|A)$.

$$\begin{aligned}\Pr(B|A) &= \frac{\Pr(A \cap B)}{\Pr(A)} \\ &= \frac{\Pr((\{H_1H_2...H_{k-1}T_k\} \cup \{T_1T_2...T_{k-1}H_k\}) \cap \{H_1\}))}{\Pr(A)} \\ &= \frac{\Pr(\{H_1H_2...H_{k-1}T_k\})}{\Pr(A)} \\ &= \boxed{\frac{p^{k-1}(1-p)}{p^{k-1}(1-p)+(1-p)^{k-1}p}}\end{aligned}$$

Equations

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

$$\Pr(B) = \sum_i \Pr(B|A_i)$$

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$p(m) = (1-p)^{m-1} p$$

$$F_Y(y) = \int_{x:g(x)\leq y} f_X(x)dx$$

$$f_Y(y) = \sum_i^n f_X(x_n)\left|\frac{dx_n}{dy}\right|$$

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x)dx = f(b(y))\frac{db(y)}{dy} - f(a(y))\frac{da(y)}{dy}$$