(15 pts) **1.** Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1].$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n-1] e^{j\left(\frac{\pi}{17}-\omega\right)n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3} e^{j\left(\frac{\pi}{17}-\omega\right)}\right)^n u[n-1] \qquad x = n-1$$

$$= \frac{1}{3} e^{j\left(\frac{\pi}{17}-\omega\right)} \sum_{x=-\infty}^{\infty} \left(\frac{1}{3} e^{j\left(\frac{\pi}{17}-\omega\right)}\right)^x u[x]$$

$\checkmark$ By (3.1)

$$\left|\frac{1}{3} e^{j\left(\frac{\pi}{17}-\omega\right)}\right| < 1$$

$$= \frac{1}{3} e^{j\left(\frac{\pi}{17}-\omega\right)} \cdot \frac{1}{1-\frac{1}{3}e^{-j\left(\frac{\pi}{17}-\omega\right)}}$$

$$X(\omega) = \frac{e^{j\left(\frac{\pi}{17}-\omega\right)}}{3-e^{-j\left(\pi/17-\omega\right)}}$$

(15 pts) **2.** Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$\mathcal{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi).$$

$$x(t) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k]\, \delta(\omega - k\pi)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \underbrace{\int_0^{2\pi} \delta(\omega - k\pi) e^{j\omega t}\, d\omega}_{= e^{jk\pi t}}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k]\, e^{jk\pi t}$$

$\omega = k$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{j\pi t}\right)^k$$

$$= \frac{1}{2\pi} \cdot \frac{1}{1 - \frac{1}{2} e^{j\pi t}}$$

$$\boxed{x(t) = \frac{1}{2\pi - \pi e^{j\pi t}}}$$

(15 pts) **3.** Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

$X(\omega) = \frac{j}{\sin\omega}$ is odd and pure imaginary
By (4.5), $x[n]$ must be real and odd.
But $x[n] = j\cos(g[n])$ is pure imaginary
Therefore, $X(\omega) = \frac{j}{\sin\omega}$ cannot be its transform

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{3}{\cos\omega}$. Could Alice be right? Explain.

$X(\omega) = \frac{3}{\cos\omega}$ is real and even. This means that $x[n]$ must be real and even by (4.4). But $x[n]$ is imaginary! So $X(\omega)$ cannot be $\frac{3}{\cos\omega}$

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{(\omega^2+1)^2}$. Could Devin be right? Explain.

$X(\omega) = \frac{j}{(\omega^2+1)^2}$ is <u>not</u> periodic or It is known that the F.T. of a discrete-time signal is always periodic.

Thus, $X(\omega) = \frac{j}{(\omega^2+1)^2}$ <u>cannot</u> be the transform of $x[n] = j\cos(g[n])$

$(2-z)(4-z)$

$8-6z+z^2$

4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

$\mathcal{F}\left(y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2]\right) = 2\,\mathcal{F}(x[n])$

by [illegible] $\rightarrow Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$

$Y(\omega)\left(1 - \frac{3}{4}(e^{-j\omega})^1 + \frac{1}{8}(e^{-j\omega})^2\right)$

$z = e^{-j\omega}$

$\frac{1}{4}(4 - 3z + \frac{1}{2}z^2)$

$\frac{1}{8}(8 - 6z + z^2)$

$\frac{1}{8}(2-z)(4-z)$

$Y(\omega)(2-e^{-j\omega})(4-e^{-j\omega})\cdot\frac{1}{8} = 2X(\omega)$

$$\frac{Y(\omega)}{X(\omega)} = \boxed{\frac{16}{(2-e^{-j\omega})(4-e^{-j\omega})} = H(\omega)}$$

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$\mathcal{F}(SYS[\delta[n]]) = H(\omega) \cdot \mathcal{F}[\delta[n]]$

$= H(\omega) \cdot 1$

Response $= \mathcal{F}^{-1}(\mathcal{F}(SYS[\delta[n]])) = \mathcal{F}^{-1}(H(\omega))$

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer.)

$X(\omega) = \mathcal{F}\left(\left(\frac{1}{4}\right)^n u[n]\right)$ $\quad \left|\frac{1}{4}\right| < 1$, And h_r (>1

$X(\omega) = \dfrac{1}{1 - \frac{1}{4} e^{-j\omega}}$

$Y(\omega) = \mathcal{F}(y[n]) = H(\omega) X(\omega)$

$= \dfrac{16}{(2 - e^{-j\omega})(4 - e^{-j\omega})(1 - \frac{1}{4} e^{-j\omega})}$

$\frac{1}{4} \cdots$

$Y(\omega) = \dfrac{4}{(2 - e^{-j\omega})(4 - e^{-j\omega})(4 - e^{-j\omega})}$

(10 pts) d) What is the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer)

Partial Fractions!

(15 pts) **5.** Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2} dt.$$

$$= \int_{-\infty}^{\infty} \pi^2 \left| \frac{\sin 4t}{\pi t} \right|^2 dt$$

$$= \pi^2 \int_{-\infty}^{\infty} \left| \frac{\sin 4t}{\pi t} \right|^2 dt$$

Parseval

$$= \pi^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \mathcal{F}\left(\frac{\sin 4t}{\pi t} \right) \right|^2 dw$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} \left| u(w+4) - u(w-4) \right|^2 dw$$

$$= \frac{\pi}{2} \int_{-4}^{4} 1 \, dw$$

$$= \frac{\pi}{2} \cdot 8$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2} dt = 4\pi}$$