Challenge problem.

I noticed something after I showed that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Recall that I used the formula

$$1 + r + r^2 + \cdots + r^N = \frac{1}{1-r} - \frac{r^{N+1}}{1-r}$$

with $r = -x^2$ to get

$$1 - x^2 + x^4 - \cdots + (-1)^N x^{2N} = \frac{1}{1+x^2} + E_N(x)$$

where $E_N(x) = \dfrac{(-1)^{N+2} x^{2(N+1)}}{1+x^2}$. Then I integrated from 0 to 1 in order to get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^N}{2N+1} = \frac{\pi}{4} + \int_0^1 E_N(x)\,dx.$$

That's when it hit me! $E_N(x)$ is

negative when N is odd and positive when N is even. So the sum $S_N = \sum_{n=0}^{N} \frac{(-1)^N}{2N+1}$ is less than $\frac{\pi}{4}$ when N is odd and greater than $\frac{\pi}{4}$ when N is even.

Could we improve our accuracy by taking an average of S_N and S_{N+1}?

I believe it improves the error estimate from $|\varepsilon| < (\text{constant})\frac{1}{N}$ to $< (\text{constant})\frac{1}{N^2}$.

Can you prove it?