HW7 Sample Solutions

9.21 (a) $P(\{S_n' = k\}) = \binom{n}{k} p'^k (1-p')^{n-k}$,

where $p' = P(\{Y_n = 1\}) = P(\{\text{not erased}\} | \text{get } 1\}) P(\{\text{get } 1\})$

$= (1-\alpha) \cdot p$

S_n' is the Bionomial counting process of $p' = (1-\alpha)p$,
thus S_n' has independent and stationary increments.

(b) $P(\{Y_n = 1\}) = P(\{\text{not erased}\} | \{\text{get } 1\}) P(\{\text{get } 1\}) +$
$P(\{\text{flip to } 1\} | \{\text{get } 0\}) P(\{\text{get } 0\})$

$= (1-\alpha)p + \beta(1-p) = p''$

S_n'' is also Bionomial counting process with p'',
thus it has indep. and stationary increments.

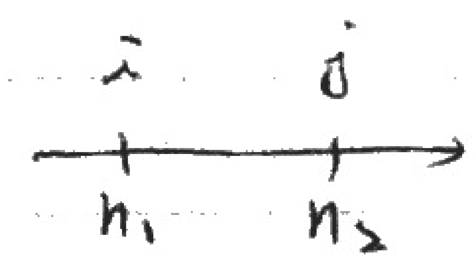
9.22 (a) $P(\{S_n = j, S_{n'} = i\}) = P(\{S_n = j, S_{n'-n} = i-j\})$ (Assume $n' > n$, $i \geq j$)

$\because$ nonoverlapping time interval $\rightarrow \overset{?}{=} P(\{S_n = j\}) P(\{S_{n'-n} = i-j\})$

But in general, $P(\{S_{n'-n} = i-j\}) \neq P(\{S_{n'} = i\})$

$\therefore P(\{S_n = j, S_{n'} = i\}) \neq P(\{S_n = j\}) P(\{S_{n'-n} = i-j\})$

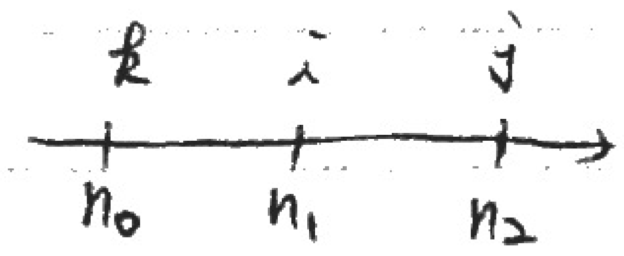
(b) $P(\{S_{n_2} = j\} | \{S_{n_1} = i\}) = \dfrac{P(\{S_{n_2} = j\} \cap \{S_{n_1} = i\})}{P(\{S_{n_1} = i\})} = \dfrac{P(\{S_{n_2} = j\} | \{S_{n_1} = i\}) \cancel{P(\{S_{n_1} = i\})}}{\cancel{P(\{S_{n_1} = i\})}}$



$= P(\{S_{n_2 - n_1} = j - i\})$

$= \binom{n_2 - n_1}{j-i} p^{j-i} (1-p)^{n_2 - n_1 - j + i}$ ※

(c) $P(\{S_{n_2} = j\} | \{S_{n_1} = i, S_{n_0} = k\}) = \dfrac{P(\{S_{n_0} = k, S_{n_1} = i, S_{n_2} = j\})}{P(\{S_{n_1} = i, S_{n_0} = k\})}$



$= \dfrac{P(\{S_{n_0} = k, S_{n_1 - n_0} = i-k, S_{n_2 - n_1} = j-i\})}{P(\{S_{n_0} = k, S_{n_1 - n_0} = i-k\})}$

nonoverlapping time interval $\rightarrow \overset{?}{=} P(\{S_{n_2 - n_1} = j - i\})$

$= P(\{S_{n_2} = j\} | \{S_{n_1} = i\})$

9.23 (a) $P(\{S_n=0\}) = P(\{\frac{n}{2}$ of 1's and $\frac{n}{2}$ of 0's$\})$

$$= \begin{cases} \binom{n}{n/2} p^{n/2} (1-p)^{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

(b) If $p=\frac{1}{2}$, $P(\{S_n=0\}) = \begin{cases} \binom{n}{n/2}\left(\frac{1}{2}\right)^n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

9.24 (a) X_n is Bernoulli R.P., $X_0=0$.

$E[X_n]=p$, $E[X_n^2]=p$

$E[Y_n] = \frac{1}{2}(E[X_n]+E[X_{n-1}]) = \underline{p}$ ※

$E[Y_n^2] = \frac{1}{4}\{E[X_n^2]+2E[X_nX_{n-1}]+E[Y_n^2]\} = \frac{1}{4}\{p+2p^2+p\} = \frac{p(p+1)}{2}$

$var(Y_n) = E[Y_n^2]-(E[Y_n])^2 = \frac{1}{2}p-\frac{1}{2}p^2$

$E[Y_nY_{n+1}] = \frac{1}{4}E[(X_n+X_{n-1})(X_{n+1}+X_n)] = \frac{1}{4}E[X_nX_{n+1}+X_n^2+X_{n-1}X_{n+1}+X_{n-1}X_n]$

$= \frac{1}{4}(p^2+p+p^2+p^2) = \frac{1}{4}(3p^2+p)$

$E[Y_nY_{n+k}] = E[Y_n]E[Y_{n+k}] = p^2$, for $k>1$

$C_Y(n,n+k) = E[Y_nY_{n+k}] - E[Y_n]E[Y_{n+k}] = E[Y_nY_{n+k}]-p^2$

$$= \begin{cases} \frac{1}{2}p-\frac{1}{2}p^2, & k=0 \quad \leftarrow var(Y_n) \\ \frac{1}{4}(p-p^2), & k=1 \\ 0, & k>1. \end{cases}$$ ※

$E[Z_n] = \frac{2}{3}E[X_n]+\frac{1}{3}E[X_{n-1}] = p$ ※

$E[Z_n^2] = \frac{1}{9}[4E[X_n^2]+4E[X_nX_{n-1}]+E[X_{n-1}^2]] = \frac{1}{9}(4p+4p^2+p) = \frac{4p^2+5p}{9}$.

$E[Z_nZ_{n+1}] = \frac{1}{9}[4E[X_nX_{n+1}]+2E[X_n^2]+2E[X_{n+1}X_{n-1}]+E[X_nX_{n-1}]] = \frac{7p^2+2p}{9}$

$E[Z_nZ_{n+k}] = E[Z_n]E[Z_{n+k}] = p^2$, for $k>1$

$$C_Z(n,n+k) = \begin{cases} \frac{5p-5p^2}{9}, & k=0 \quad \leftarrow Var(Z_n) \\ \frac{2p-2p^2}{9}, & k=1 \\ 0, & k>1. \end{cases}$$ ※

(b) X_n is Random step process.

$E[X_n] = 1\cdot p+(-1)(1-p) = 2p-1$

$E[X_n^2] = 1^2\cdot p+(-1)^2(1-p) = 1$

(9.24 (b) continued)

Same procedure as in (a), one can get

$E[Y_n] = 2p-1$ *

$E[Y_n^2] = \frac{1}{2} + \frac{(2p-1)^2}{2}$

$E[Y_n Y_{n+1}] = \frac{1}{4} + \frac{3}{4}(2p-1)^2$

$E[Y_n Y_{n+k}] = (2p-1)^2$, for $k>1$

$$\therefore C_Y(n, n+k) = \begin{cases} \frac{1}{2} - \frac{1}{2}(2p-1)^2 & , k=0 \leftarrow \text{var}(Y_n) \\ \frac{1}{4} - \frac{1}{4}(2p-1)^2 & , k=1 \\ 0 & , k>1. \end{cases}$$ *

$E[Z_n] = 2p-1$ *

$E[Z_n^2] = \frac{5}{9} + \frac{4}{9}(2p-1)^2$

$E[Z_n Z_{n+1}] = \frac{2}{9} + \frac{7}{9}(2p-1)^2$

$E[Z_n Z_{n+k}] = (2p-1)^2$

$$\therefore C_Z(n, n+k) = \begin{cases} \frac{5}{9} - \frac{5}{9}(2p-1)^2 & , k=0 \\ \frac{2}{9} - \frac{2}{9}(2p-1)^2 & , k=1 \\ 0 & , k>1 \end{cases}$$ *

9.26 (a) $E[M_n] = \frac{1}{n}[E[X_1] + \cdots + E[X_n]] = \frac{1}{n} \cdot n E[X] = E[X].$

$$\begin{aligned} C_M(n, k) &= E[(M_n - E[M_n])(M_k - E[M_k])] \\ &= E[(\tfrac{1}{n}S_n - E[X])(\tfrac{1}{k}S_k - E[X])] = E[\tfrac{1}{n}(S_n - nE(X))\tfrac{1}{k}(S_k - kE(X))] \\ &= \frac{1}{nk} E[(S_n - E(S_n))(S_k - E[S_k])] \\ &= \frac{1}{nk} C_S(n,k) = \frac{1}{nk} \min(n,k)\,\sigma_X^2. \end{aligned}$$

(Last eq. follows from textbook p. 506).

$$\text{var}(M_n) \underset{n=k}{=} C_M(n,n) = \frac{1}{n^2} \cdot n\,\sigma_X^2 = \frac{1}{n}\sigma_X^2.$$

(b) $$\begin{aligned} M_{n+1} - M_n &= \frac{X_1 + \cdots + X_n + X_{n+1}}{(n+1)} - \frac{X_1 + \cdots + X_n}{n} \\ &= \frac{n(X_1 + \cdots + X_{n+1}) - (n+1)(X_1 + \cdots + X_n)}{n(n+1)} = \frac{1}{n+1}X_{n+1} - \frac{1}{n+1}M_n \end{aligned}$$

$\therefore$ M_n does <u>not</u> have independent increments.

9.34 $\lambda = 10.$ $t = \frac{1}{3}.$

$$P(N(\tfrac{1}{3}) = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\frac{10}{3}}$$ *

9.35 Suppose you get the item with probability p after inserting \$1.
Let $N(t)$ be amount of money inserted before time t.
Let $D(t)$ be number of dispensed item.

$$P(\{D(t)=k\} \mid \{N(t)=n\}) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

By total prob. law,

$$P(\{D(t)=k\}) = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

$$\overset{(n'=n-k)}{=} \sum_{n'=0}^{\infty} \frac{1}{k!\,n'!} p^k (1-p)^{n'} \cdot (\lambda t)^{n'} \cdot (\lambda t)^k \cdot e^{-\lambda t}$$

$$= \frac{e^{-\lambda t} (p\lambda t)^k}{k!} \sum_{n'=0}^{\infty} \frac{[\lambda t (1-p)]^{n'}}{n'!}$$

(Recall: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$)

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t p}$$

※

9.36 (a) $P(\{N(t)=0\}) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$ ※

(b) $P(\{$ message can not be corrected $\}) = P(\{$ more than 2 impulses in t seconds $\})$

$$= P(\{N(t)>2\}) = 1 - P(\{N(t)=0\}) - P(\{N(t)=1\}) - P(\{N(t)=2\})$$

$$= 1 - e^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2!}\right)$$

※

9.38 (a) $P(\{N(t-d)=j\} \mid \{N(t)=k\})$

$$= \frac{P(\{N(t-d)=j\})\, P(\{N(t)-N(t-d)=k-j\})}{P(\{N(t)=k\})}$$

$$= \frac{\frac{\lambda^j (t-d)^j}{j!} e^{-\lambda(t-d)} \cdot \frac{\lambda^{k-j} d^{k-j}}{(k-j)!} e^{-\lambda d}}{\frac{\lambda^k t^k}{k!} e^{-\lambda t}}$$

$$= \frac{k!}{j!(k-j)!} \left(\frac{t-d}{t}\right)^j \left(\frac{d}{t}\right)^{k-j}$$

※

9.38 (b) $P(\{N(t+d)=j\} \mid \{N(t)=k\}) = \dfrac{\cancel{P(\{N(t)=k\})}\,P(\{N(t+d)-N(t)=j-k\})}{\cancel{P(\{N(t)=k\})}}$

$$= \frac{(\lambda d)^{j-k} \cdot e^{-\lambda d}}{(j-k)!} \quad ※$$

In part (a), the conditional pmf becomes a Binomial pmf with $p=(1-\frac{d}{t})$.
In part (b), it's a Poisson pmf.

9.40 (a) Let $N(t)$ be number of customers arrived in time t.

$$P(\{N(t)=k\} \mid \{T=t\}) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$P(\{N(t)=k\}) = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} f_T(t)\, dt \quad ※$$

(b) Let $f_T(t) = \beta e^{-\beta t}$, $t \geq 0$.

$$P(\{N(t)=k\}) = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \beta e^{-\beta t}\, dt$$

$$= \frac{\beta}{k!} \int_0^\infty (\lambda t)^k \cdot e^{-(\lambda+\beta)t}\, dt$$

$(t'=(\lambda+\beta)t)$ $$= \frac{\beta}{k!} \cdot \frac{\lambda^k}{(\lambda+\beta)^k} \int_0^\infty t'^k e^{-t'} \frac{dt'}{(\lambda+\beta)}$$

$$= \frac{\beta}{k!} \frac{\lambda^k}{(\lambda+\beta)^{k+1}} \cdot \underbrace{\Gamma(k+1)}_{=k!}$$ (see textbook p.170) properey of Gamma fcn.

$$= \left(\frac{\beta}{\lambda+\beta}\right)\left(\frac{\lambda}{\lambda+\beta}\right)^k, \quad k=0,1,2,\ldots \quad ※$$

9.42 (a) First, we know $N(t) = N_1(t) + N_2(t)$

Thus, $P = P(\{\text{get } j \text{ heads in } (k+j) \text{ tosses}\})$

$$= \binom{k+j}{j} p^j (1-p)^k \quad ※$$

(b) $P(\{N_1(t)=j, N_2(t)=k\}) = P(\{N_1(t)=j, N_2(t)=k\} / \{N(t)=j+k\}) \cdot P(\{N(t)=j+k\})$

$$= \binom{j+k}{j} p^j (1-p)^k \cdot \frac{(\lambda t)^{j+k}}{(j+k)!} e^{-\lambda t}$$

$$= \underline{\frac{(p\lambda t)^j}{j!} \cdot e^{-p\lambda t}} \cdot \underline{\frac{[(1-p)\lambda t]^k}{k!} \cdot e^{-(1-p)\lambda t}} \quad ※$$

9.43 (a) Let X_i be the random reward got from the i-th time

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

$$P(\{X(t)=j\}) = \sum_{n=0}^{\infty} P(\{X(t)=j\} \mid \{N(t)=n\})\, P(\{N(t)=n\})$$

$$= \binom{n}{j} p^j (1-p)^{n-j} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

(from 9.35) $$= \frac{(\lambda p t)^j}{j!} e^{-\lambda p t}, \quad j = 0, 1, \ldots$$

(b) $X_i = \{0, 5\}$,

$P(\{X_i = 0\}) = \frac{5}{6}$, $P(\{X_i = 5\}) = \frac{1}{6}$

$P(\{X_x = 5j\}) = \frac{(\lambda p t)^j}{j!} e^{-\lambda p t}$ with $p = \frac{1}{6}$, $j = 0, 1, 2, \ldots$

(c) $P(\{X(t)=j\} \mid \{N(t)=n\}) = \frac{n^j e^{-n}}{j!}$,

because $X(t) = \sum_{i=1}^{n} X_i$ is the sum of n iid Poisson RV with mean $n \cdot E(X_i) = n$.

$$P(\{X(t)=j\}) = \sum_{n=0}^{\infty} \frac{n^j}{j!} e^{-n} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= \frac{e^{-\lambda t}}{j!} e^{\frac{\lambda t}{e}} \sum_{n=0}^{\infty} n^j \cdot \frac{\left(\frac{\lambda t}{e}\right)^n}{n!} \cdot e^{-\left(\frac{\lambda t}{e}\right)}$$

$$= \frac{e^{-\lambda t + \frac{\lambda t}{e}}}{j!} E[N^j]$$ where N is Poisson rate with parameter $\frac{\lambda}{e}$.

(d) $P(\{X(t)=j\} \mid \{N(t)=n\}) = \underset{\uparrow}{p}\, \underline{(1-p)^{n-j}} \longrightarrow$ rest of the $(n-j)$ trials must not return

the j-th trial must return all coins

$$P(\{X(t)=j\}) = \sum_{n=j}^{\infty} p(1-p)^{n-j} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= p(\lambda t)^j e^{-\lambda t} \sum_{n'=0}^{\infty} \frac{[(1-p)\lambda t]^{n'}}{(n'+j)!}$$ ※