9

(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{2j}\right)^{|n|}.$$

$x[n] = \left(\frac{1}{2j}\right)^{n}$

By 2.5: $\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{|n|} e^{-j\omega n}$

$\left(\frac{1}{2j}\right)^0 = 1$ $\qquad = 1 + 2\sum_{n=1}^{\infty} \left(\frac{e^{-j\omega}}{2j}\right)^{n}$ $\qquad r = n-1$, $n = r+1$

$= 1 + 2\sum_{r=0}^{\infty} \left(\frac{e^{-j\omega}}{2j}\right)^{r+1}$

$= 1 + \frac{e^{-j\omega}}{j} \sum_{r=0}^{\infty} \left(\frac{e^{-j\omega}}{2j}\right)^{r}$

$= 1 + \frac{e^{-j\omega}}{j} \cdot \frac{1}{1 - \frac{e^{-j\omega}}{2j}}$

$= \begin{cases} \left(\frac{1}{2j}\right)^{n} & n \geq 0 \\ \left(\frac{1}{2j}\right)^{-n} & n < 0 \end{cases}$

$\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

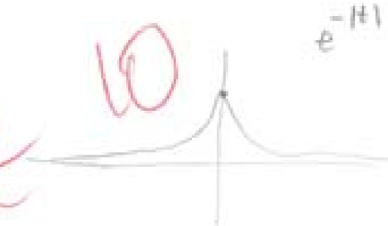
$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{|n|} e^{-j\omega n}$

$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2j}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n}$

$= \sum_{n=-\infty}^{-1} \left(\frac{e^{j\omega}}{2j}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2je^{j\omega}}\right)^{n}$

Let $k = -n$.

$= \sum_{k=1}^{\infty} \left(\frac{e^{j\omega}}{2j}\right)^{k} + \sum_{n=0}^{\infty} \left(\frac{1}{2je^{j\omega}}\right)^{n} = \frac{1}{1 - \frac{e^{j\omega}}{2j}} - 1 + \frac{1}{1 - \frac{1}{2je^{j\omega}}}$



10

(20 pts) **2.** The Frequency response of a continuous-time LTI system is

$$H(j\omega) = \mathcal{H}(\omega) = \frac{1}{j\omega + 2}.$$

-2

Use the convolution property of the Fourier transform to determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$.

By: 1 $\quad X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t}\, dt$

$= 2\int_0^{\infty} e^{-t} e^{-j\omega t}\, dt$

$= 2\int_0^{\infty} e^{-(j\omega+1)t}\, dt$

$= \frac{-2}{j\omega+1} e^{-(j\omega+1)t}\Big|_0^{\infty}$

$= \frac{-2}{j\omega+1}\cdot(0-1) = \cancel{\frac{2}{j\omega+1}}$

$X(\omega) = F(e^{-t}u(t) + e^{t}u(-t))$

by 13: $= F(e^{-t}u(t)) + F(u^{-t}u(t))$

$= \frac{1}{1+j\omega} + \frac{1}{1-j\omega}$ by 7

-4 -2 -2

but $\quad H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad \therefore Y(\omega) = H(\omega)X(\omega) = \frac{1}{j\omega+2}\cdot\frac{2}{j\omega+1}$

$= \frac{A}{j\omega+2} + \frac{B}{j\omega+1}$

$Aj\omega + A + Bj\omega + 2B = 2$

$A+2B = 2$ $\quad B = 2$

$A+B = 0$ $\quad A = -2$

$= \frac{-2}{j\omega+2} + \frac{2}{j\omega+1}$

$y(t) = F^{-1}(Y(\omega)) = F^{-1}\left(\frac{-2}{j\omega+2} + \frac{2}{j\omega+1}\right) \overset{\text{By 9}}{=} -2F^{-1}\left(\frac{1}{j\omega+2}\right) + 2F^{-1}\left(\frac{1}{j\omega+1}\right)$

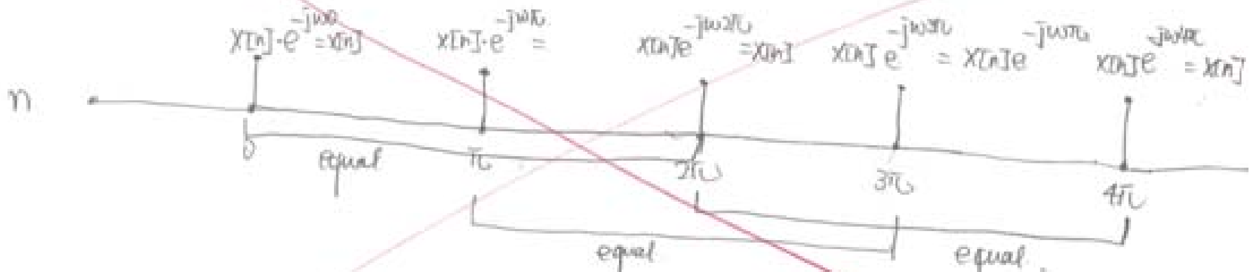
By 7 $\boxed{-2e^{-2t}u(t) + 2e^{-t}u(t)}$

5

(15 pts) **3.** True/False? The Fourier transform of a DT signal $x[n]$ is a periodic function, no matter what $x[n]$ is. (Justify your answer.)

By 25: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

F.T of DT signal ~~x[n]~~ is periodic function with period 2π b/c



~~it's repeating by period of 2π~~

No, $X(\omega)$ is a DT fct.

The F.T of a DT signal is periodic with period 2π b/c it is a linear combination of functions which are periodic with period 2π.

$$X(\omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi)n}$$
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j2\pi n} \quad \uparrow 1$$
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$
$$= X(\omega)$$

10

(10 pts) **4.** A continuous-time LTI system has frequency response

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}.$$

Derive a differential equation representing this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(j\omega) = \cancel{H}(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 4}{(j\omega+2)(j\omega+3)}$$

$$X(\omega)(j\omega+4) = Y(\omega)(j\omega+2)(j\omega+3)$$

$$j\omega X(\omega) + 4X(\omega) = (j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega)$$

$$\mathcal{F}^{-1}(j\omega X(\omega) + 4X(\omega)) = \mathcal{F}^{-1}((j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega))$$

By: 9 $\quad \mathcal{F}^{-1}(j\omega X(\omega)) + 4\mathcal{F}^{-1}(X(\omega)) = \mathcal{F}^{-1}((j\omega)^2 Y(\omega)) + 5\mathcal{F}^{-1}(j\omega Y(\omega)) + 6\mathcal{F}^{-1}(Y(\omega))$

By 16: $\quad \frac{dx(t)}{dt} + 4x(t) = \frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t)$

6

(10 pts) **5.** A CT signal $x(t)$ has Fourier transform

$$\mathcal{X}(\omega) = -2e^{(j-1)\omega}u(\omega+1).$$

Denote by $y(t)$ the signal obtained by delaying $x(t)$ by six seconds. Sketch a graph representing the magnitude $|\mathcal{Y}(\omega)|$ of the Fourier transform $\mathcal{Y}(\omega)$ of $y(t)$. (Justify your answer.)

$y(t) = x(t-6)$.

By 10: $Y(\omega) = \mathcal{F}(y(t)) = \mathcal{F}(x(t-6)) = e^{-6j\omega} X(\omega)$

$= e^{-6j\omega} \cdot (-2e^{(j-1)\omega} u(\omega+1))$

$= -2e^{(j-1-6j)\omega} u(\omega+1)$

$$|Y(\omega)| = \begin{cases} 2|e^{(j-1-6j)\omega}| = 2e^{-\omega}, & \omega \geq -1 \\ 0, & \omega < -1 \end{cases}$$