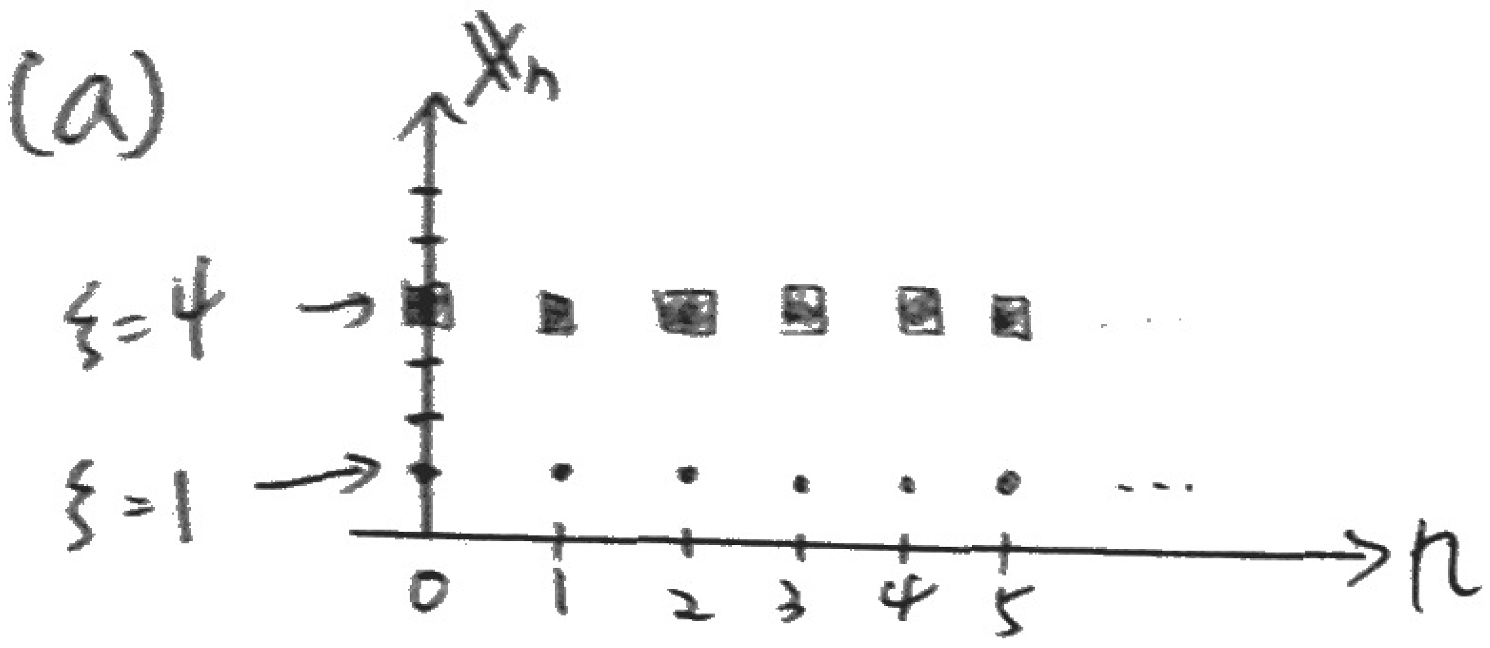
HW6 Sample Solution

9.2 (a)



ξ is the outcome of tossing a fair die

(b) $P(X_n = k) = \frac{1}{6}$, $k = \{1,2,3,4,5,6\}$

$P(X_n = k) = 0$, $\forall k \neq \{1,2,3,4,5,6\}$.

(c) $P(X_n = 1, X_{n+k} = 1) = \frac{1}{6}$

$P(X_n = 2, X_{n+k} = 2) = \frac{1}{6}$

$\vdots \qquad \vdots$

$P(X_n = 6, X_{n+k} = 6) = \frac{1}{6}$

$P(X_n = k_1, X_{n+k} = k_2) = 0$, $k_1 \neq k_2$

$\therefore P(X_n = k_1, X_{n+k} = k_2)$

$$= \begin{cases} \frac{1}{6}, & k_1 = k_2 \in \{1,2,3,4,5,6\} \\ 0, & k_1 \neq k_2 \\ 0, & k_1, k_2 \notin \{1,2,3,4,5,6\} \end{cases}$$

\#

(d) $E[X_n] = \sum_{k=1}^{6} k \cdot P_X(k) = (1+2+\cdots+6)\cdot\frac{1}{6} = \frac{7}{2}$ \#

$C_X(n, n+k) = E[X_n X_{n+k}] - E[X_n]E[X_{n+k}]$

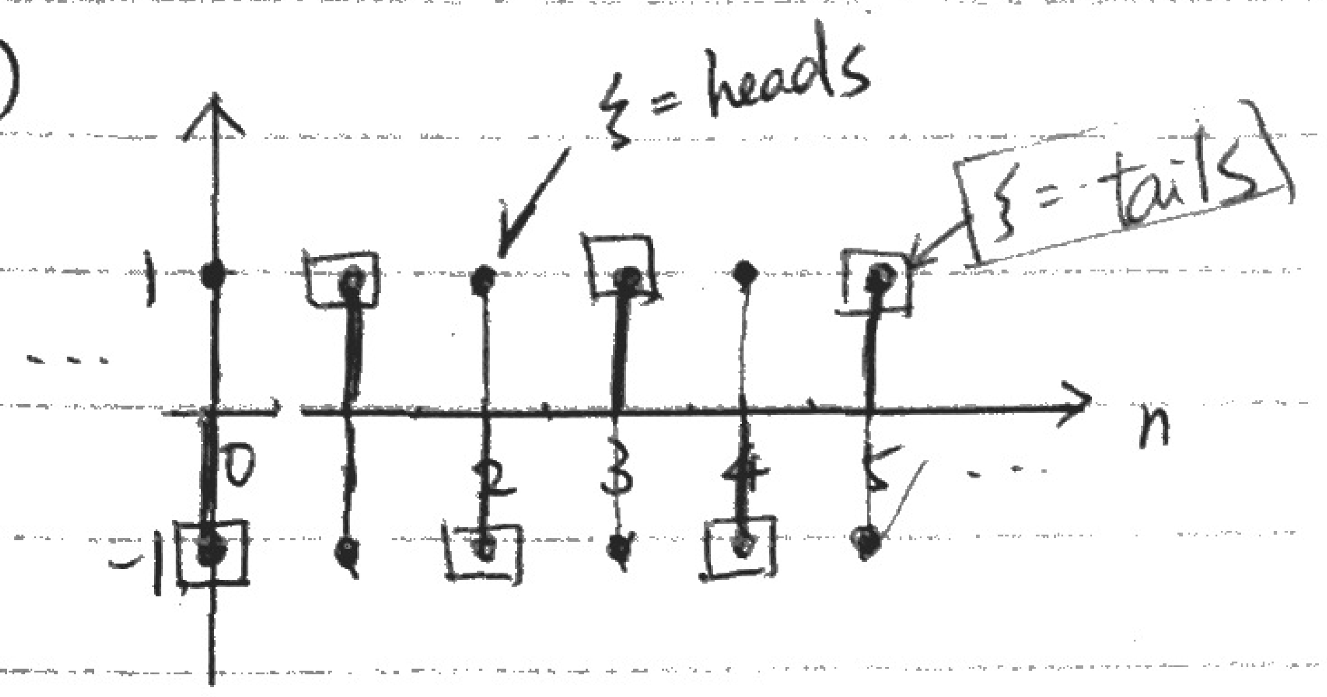
$$E[X_n X_{n+k}] = \sum_{k_1=1}^{6}\sum_{k_2=1}^{6} k_1 k_2 P(X_n = k_1, X_{n+k} = k_2)$$

$$= 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + 3^2\cdot\frac{1}{6} + 4^2\cdot\frac{1}{6} + 5^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{6} = \frac{91}{6}$$

($\because P(X_n = k_1, X_{n+k} = k_2) = 0$ for $k_1 \neq k_2$)

$\therefore C_X(n, n+k) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$ \#

9.3 (a)



$\xi = \{\text{heads, tails}\}$.

(b) $P(X_n = 1) = \frac{1}{2}$

$P(X_n = -1) = \frac{1}{2}$

(c) k is even, $P(X_n = 1, X_{n+k} = 1) = P(X_n = -1, X_{n+k} = -1) = P(\{\xi = \text{heads}\}) = \frac{1}{2}$

$P(X_n = \pm 1, X_{n+k} = \mp 1) = 0$.

k is odd, $P(X_n = 1, X_{n+k} = -1) = P(X_n = -1, X_{n+k} = 1) = \frac{1}{2}$

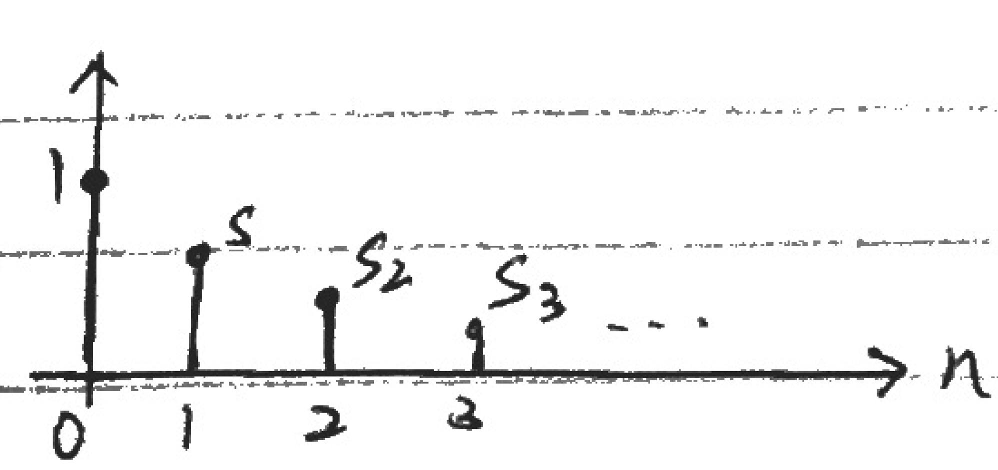
$P(X_n = \pm 1, X_{n+k} = \pm 1) = 0$

(d) $E[X_n] = 1 \cdot (\frac{1}{2}) + (-1)(\frac{1}{2}) = 0$

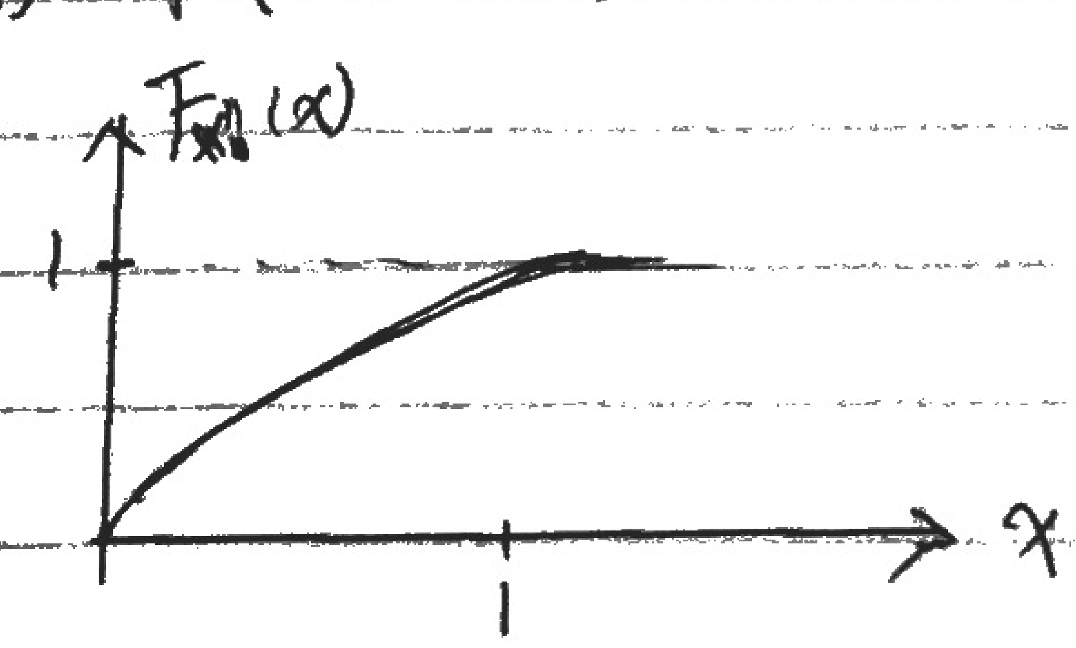
$$E[X_n X_{n+k}] = \begin{cases} 1^2 \cdot (\frac{1}{2}) + (-1)^2 \cdot \frac{1}{2} = 1, & k \text{ even} \\ 1 \cdot (-1)(\frac{1}{2}) + (-1) \cdot 1 (\frac{1}{2}) = -1, & k \text{ odd} \end{cases}$$

$$\therefore C_X(n, n+k) = \begin{cases} 1, & k \text{ is even} \\ -1, & k \text{ is odd} \end{cases} \ast$$

9.4 (a) $X_n = s^n$, $s \in (0,1)$



(b) $F_{X_n}(x) = P(s^n \le x) = P(s \le x^{1/n}) = x^{1/n}$, $\because s \sim U([0,1))$.



(c) For $0 \le x_1, x_2 < 1$,

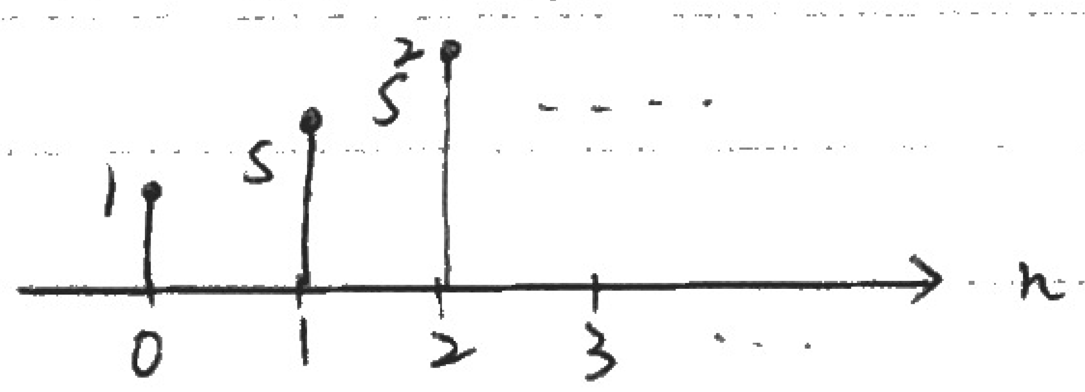
$$P(X_n \le x_1, X_{n+1} \le x_2) = P(s^n \le x_1, s^{n+1} \le x_2)$$
$$= P(s \le x_1^{1/n}, s \le x_2^{1/n+1})$$
$$= P(s \le \min\{x_1^{1/n}, x_2^{1/n+1}\})$$
$$= \min\{x_1^{1/n}, x_2^{1/n+1}\}. \quad \#$$

(d) $$E[X_n] = E[s^n] = \int_0^1 s^n \, ds = \frac{1}{n+1} s^{n+1} \Big|_0^1 = \frac{1}{n+1} \quad \#$$

$$E[X_n X_{n+k}] = E[s^n \cdot s^{n+k}] = E[s^{2n+k}]$$
$$= \int_0^1 s^{2n+k} \, ds = \frac{1}{2n+k+1}$$

$$\therefore C_X(n, n+k) = \frac{1}{2n+k+1} - \left(\frac{1}{n+1}\right)\left(\frac{1}{n+k+1}\right) \quad \#$$

(e) $s \sim U((1,2))$.



$$P(X_n \le x) = P(s^n \le x) = P(s \le x^{1/n}) = P(s-1 \le x^{1/n} - 1)$$

(Since $s \sim U((1,2))$, $s-1 \sim U((0,1))$) $= x^{1/n} - 1$.

$$P(X_n \le x_1, X_{n+1} \le x_2) = P(s^n \le x_1, s^{n+1} \le x_2)$$
$$= P(s \le x_1^{1/n}, s \le x_2^{1/n+1})$$
$$= P(s-1 \le x_1^{1/n} - 1, s-1 \le x_2^{1/n+1} - 1)$$
$$= P(s-1 \le \min\{x_1^{1/n} - 1, x_2^{1/n+1} - 1\})$$
$$= \min(x_1^{1/n}, x_2^{1/n+1}) - 1$$

$$E[X_n] = E[s^n] = \int_1^2 s^n \, ds = \frac{1}{n+1} s^{n+1} \Big|_1^2 = \frac{2^{n+1} - 1}{n+1}$$

$$C_X(n, n+k) = \frac{2^{2n+k-1}}{2n+k+1} - \left(\frac{2^{n+1} - 1}{n+1}\right)\left(\frac{2^{n+k+1} - 1}{n+k+1}\right) \quad \#$$

$\because A = \pm 1$ with prob. $\frac{1}{2}$.

9.5 (a) $P(X(t) = 1) = P(X(t) = -1) = \frac{1}{2}$, $t \in [0,1]$

$P(X(t) = 0) = 1$, $t \notin [0,1]$. #

(b) $m_X(t) = \begin{cases} 1 \cdot (\frac{1}{2}) + (-1)(\frac{1}{2}) = 0 & , t \in [0,1] \\ 0 & , \text{o.w.} \end{cases}$ #

(c) We should discuss 3 cases which

① $t+d \in [0,1]$, $t \in [0,1]$

② $t \in [0,1]$, $t+d \notin [0,1]$

③ $t \notin [0,1]$

For ①, $P(X(t) = \pm 1, X(t+d) = \pm 1) = \frac{1}{2}$.

$P(X(t) = \pm 1, X(t+d) = \mp 1) = 0$

For ②, $P(X(t) = \pm 1, X(t+d) = 0) = \frac{1}{2}$

For ③, $P(X(t) = 0, X(t+d) = 0) = 1$.

(d) $C_X(t, t+d) = E[X(t) \cdot X(t+d)] - \cancelto{0}{E[X(t)]}\, \cancelto{0}{E[X(t+d)]}$

$= E[X(t) X(t+d)]$

$= \begin{cases} 1 & , t \in [0,1], t+d \in [0,1] \\ 0 & , \text{o.w.} \end{cases}$