1. Let g_n and f_n be measurable functions converging a.e. to g, f respectively. Suppose also $g, g_n \in L^1, \int g_n \to \int g$, and $g_n \geq |f_n|$. Then, $\int f_n \to \int f$.

2. Let f_n be measurable functions on (X, Υ, μ). Show f_n converges in measure $\Leftrightarrow$ f_n is Cauchy in measure. (We say f_n is Cauchy in measure $\Leftrightarrow \forall \epsilon > 0, \exists N$ such that $n \geq m \geq N \Rightarrow |\{|f_n - f_m| \geq \epsilon\}| < \epsilon$.)

3. Prove or disprove there exists $f \in L^p(0,1), 0 < p < \infty$ but $f \notin L^\infty(0,1)$.

4. Let A and B be two subsets of $\mathbb{R}$. Define $A+B = \{a+b : a \in A, b \in B\}$.

   (a) Suppose A is closed and B is compact. Prove that $A+B$ is closed.

   (b) Give an example where A and B are closed but $A+B$ is not closed.

5. Given $(X, \mathcal{M}, \mu), 1 \leq p \leq \infty, 0 < q < p$. If $f_n \to f$ and $g_n \to g$ in L^p, show that
$$\lim_{n\to\infty} \int_X |f_n|^{p-q}|g_n|^q d\mu = \int_X |f|^{p-q}|g|^q d\mu.$$

6. Suppose that f is a real-valued function defined on I. Show that if f is not constant and $f' = 0$ a.e., then f cannot be Lipschitz on I.

7. Let $f, g \in L^p$. Show $f + g \in L^p$ for all $0 < p \leq \infty$.

8. Let C be the cantor set, and for all $x \in [0,1]$ let $x = .x_1x_2x_3...$ the unique ternary expansion which lets $C = \{x : x_i = 0 \text{ or } 2\}$. (Careful here, $1/3 \in C$ so $1/3 = .0\overline{22}$, not .1.) Next, let $\phi : C \to [0,1]$ by using the binary representation for elements of $[0,1]$ and letting ϕ change all the 2s to 1s; so, for example, $\phi(.002202) = .001101$. Show the following:

   (a) ϕ is surjective but not injective.

   (b) ϕ is continuous.

   (c) Let $\mathcal{M}$ be the Lebesgue measurable sets on $[0,1]$, and so $\phi^{-1}(\mathcal{M})$ is a sigma algebra of sets in C. Let $\mu : \phi^{-1}(\mathcal{M}) \to \mathbb{R}_{\geq 0}$ by $\mu(A) = |\phi(A)|$. Show μ is a measure on C.

9. (a) Show convergence in $L^p \Rightarrow$ convergence in measure, but not conversely.

   (b) If f_n and f are in L^p with f_n converging in measure to f, does this imply $||f_n||_p \to ||f||_p$?