Fourier Series Examples

Ex: $x(t) = \sin(\pi t) + 2\cos(2\pi t)$ Find a_k

$T = 2$

$x(t+T) = \sin(\pi t + \pi T) + 2\cos(2\pi t + 2\pi T)$

for $x(t+T) = x(t)$, $\quad \pi T = k_1 (2\pi)$ $\qquad k_1, k_2 \in \mathbb{Z}$

$2\pi T = k_2 (2\pi)$

$T = 2k_1 \quad \& \quad T = k_2$

The smallest possible value of T is for $k_1 = 1 \Rightarrow T = 2$

The normal way to find a_k is to use

$$a_k = \frac{1}{T}\int_T x(t)\, e^{j\frac{2\pi}{T}t}\, dt$$

but when $x(t)$ is something with complex exponentials ($e^{j\frac{2\pi}{T}t}$, $\cos(\frac{2\pi}{T}t)$, $\sin(\frac{4\pi}{T}t)$, etc), it WILL NOT WORK!

observation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$ and we want to find a_k's

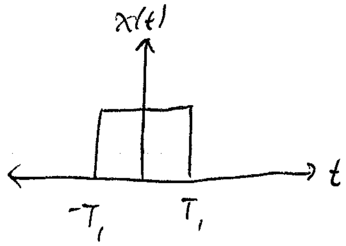
but $$x(t) = \frac{1}{2j} e^{j\pi t} - \frac{1}{2j} e^{-j\pi t} + e^{j2\pi t} + e^{-j2\pi t}$$

So, we can actually just "read off" the a_k's!

$$a_k = \begin{cases} 1 & k = \pm 2 \\ \frac{1}{2j} & k = 1 \\ -\frac{1}{2j} & k = -1 \\ 0 & \text{o.w.} \end{cases}$$

/Ex

Ex



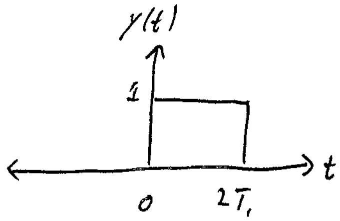
$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & ow \end{cases}$$ and periodic with period T $(T_1 < T)$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} (1)\, e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{T}\left(\frac{-T}{j2\pi k}\right) e^{-j\frac{2\pi}{T}kt}\Big|_{-T_1}^{T_1}$$

$$= \frac{-1}{j2\pi k}\left(e^{-j\frac{2\pi}{T}kT_1} - e^{j\frac{2\pi}{T}kT_1}\right)$$

$$= \frac{-1}{j2\pi k}\left(-2j\sin\left(\frac{2\pi T_1}{T}k\right)\right) = \frac{\sin\left(\frac{2\pi T_1}{T}k\right)}{\pi k}$$ "sinc function"

Now, let $y(t)$ be



Options:

1) Compute b_k from scratch.
2) Recognize $y(t)$ is a time-shifted version of $x(t)$ and use the property

$$y(t) = x(t - T_1)$$

$$b_k = e^{-jk\frac{2\pi}{T}T_1} a_k = e^{-jk2\pi\frac{T_1}{T}} \frac{\sin\left(\frac{2\pi T_1}{T}k\right)}{\pi k}$$

/Ex

Ex: $x[n] = e^{j\frac{7\pi n}{5}}$ Find a_k

First, we need to find the period, keeping in mind the period must be an integer.

$$x[n+N] = e^{j\frac{7\pi}{5}(n+N)} = e^{j\frac{7\pi}{5}N} e^{j\frac{7\pi}{5}n}$$

$$x[n+N] = x[n] \iff \frac{7\pi}{5}N = 2\pi k_1 \qquad k_1 \in \mathbb{Z}$$

$$N = \frac{10}{7}k_1 \Rightarrow N = 10$$

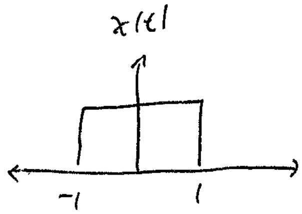
Now we use the same trick as for the continuous-time case:

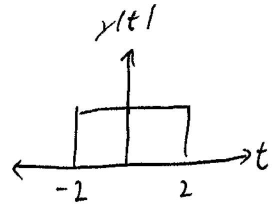
$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{10}kn} = \sum_{k=0}^{N-1} a_k e^{j\frac{\pi}{5}kn}$$

So, $a_k = \begin{cases} 1 & k=7 \\ 0 & \text{ow} \end{cases}$

/Ex

Ex





both have period 10

Let $z(t) = x(t) * y(t)$

Find $c_k \overset{FS}{\Longleftrightarrow} z(t)$

Let $a_k \overset{FS}{\longleftrightarrow} x(t)$

$b_k \overset{FS}{\longleftrightarrow} y(t)$

Then $c_k = a_k b_k$

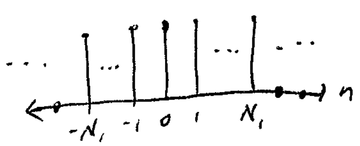
$$a_k = \frac{\sin\left(\frac{2\pi(1)}{10}k\right)}{\pi k} \qquad b_k = \frac{\sin\left(\frac{2\pi(2)}{10}k\right)}{\pi k}$$

from a previous example

$$c_k = a_k b_k = \frac{\sin\left(\frac{\pi}{5}k\right)\sin\left(\frac{2\pi}{5}k\right)}{(\pi k)^2}$$

/Ex

Ex Let $x[n]$



where $N_1 < N$

Find a_k

$$a_k = \sum_{n=-N_1}^{N_1} (1)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N_1} e^{-j\frac{2\pi}{N}kn} + \sum_{n=-N_1}^{0} e^{-j\frac{2\pi}{N}kn} - e^{-j\frac{2\pi}{N}kn}\Big|_{n=0}$$

$$= \sum_{n=0}^{N_1} \left(e^{-j\frac{2\pi}{N}k}\right)^n + \sum_{\ell=0}^{N_1} \left(e^{j\frac{2\pi}{N}k}\right)^\ell - 1$$

$$= \frac{1-\left(e^{-j\frac{2\pi}{N}k}\right)^{N_1+1}}{1-e^{-j\frac{2\pi}{N}k}} + \frac{1-\left(e^{j\frac{2\pi}{N}k}\right)^{N_1+1}}{1-e^{+j\frac{2\pi}{N}k}} - 1$$

$$= \frac{1-e^{-j\frac{2\pi}{N}k(N_1+1)}}{e^{-j\frac{\pi}{N}k}\left(e^{j\frac{\pi}{N}k}-e^{-j\frac{\pi}{N}k}\right)} + \frac{1-e^{j\frac{2\pi}{N}k(N_1+1)}}{e^{j\frac{\pi}{N}k}\left(e^{-j\frac{\pi}{N}k}-e^{j\frac{\pi}{N}k}\right)} - 1$$

$$= \frac{e^{j\frac{\pi}{N}k}-e^{j\frac{\pi}{N}k}e^{-j\frac{2\pi}{N}k(N_1+1)}}{2j\sin\left(\frac{\pi}{N}k\right)} + \frac{e^{-j\frac{\pi}{N}k}-e^{-j\frac{\pi}{N}k}e^{j\frac{2\pi}{N}k(N_1+1)}}{-2j\sin\left(\frac{\pi}{N}k\right)} - 1$$

$$= \frac{1}{2j\sin\left(\frac{\pi}{N}k\right)}\left[e^{j\frac{\pi}{N}k}-e^{-j\frac{\pi}{N}k}+e^{-j\frac{\pi}{N}k}e^{j\frac{2\pi}{N}k(N_1+1)}-e^{j\frac{\pi}{N}k}e^{-j\frac{2\pi}{N}k(N_1+1)}\right] - 1$$

$$= \frac{2j\sin\left(\frac{\pi}{N}k\right)}{2j\sin\left(\frac{\pi}{N}k\right)} + \frac{2j\sin\left(-\frac{\pi}{N}k+\frac{2\pi}{N}k(N_1+1)\right)}{2j\sin\left(\frac{\pi}{N}k\right)} - 1$$

$$= \frac{\sin\left(\frac{2\pi k N_1}{N}+\frac{\pi k}{N}\right)}{\sin\left(\frac{\pi}{N}k\right)} \qquad \text{"discrete-time sinc"}$$

/Ex