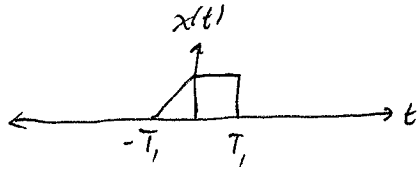
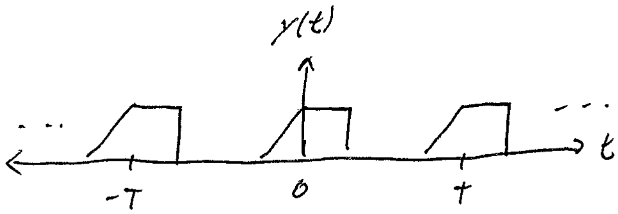
Continuous-Time Fourier Transform

Let $x(t)$ be a finite-duration signal (non-periodic)



Let $y(t)$ be the same signal replicated with period T.



We know that $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{2\pi}{T}t}$ & $b_k = \frac{1}{T}\int_T y(t) e^{-jk\frac{2\pi}{T}t}\,dt$

So, $b_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) e^{-jk\frac{2\pi}{T}t}\,dt$

$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\frac{2\pi}{T}t}\,dt$ since $y(t) = x(t)$ for $t \in [-\frac{T}{2}, \frac{T}{2}]$

Let $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$

So that $b_k = \frac{1}{T} X(jk\frac{2\pi}{T})$

Now, $y(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\frac{2\pi}{T}) e^{jk\frac{2\pi}{T}t}$ Let $\omega_0 = \frac{2\pi}{T}$

$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t}\,\omega_0$

As we let $T \to \infty$, pushing the replicas out until they effectively no longer exist, $\omega_0 \to 0$

Remember Reimann Sum: $\lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} f(k\Delta t)\Delta t = \int_{-\infty}^{\infty} f(t)\,dt$

So, we get

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\,dt \\ X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt \end{aligned}}$$

The Fourier Transform can be taken of non-periodic signals.
It looks and functions very similarly to the Fourier Series, with the exception that you use integrals rather than summations.

Ex: $x(t) = e^{-|t|}$ Find $X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-|t|}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-t}\, e^{-j\omega t}\, dt + \int_{-\infty}^{0} e^{t}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-t(1+j\omega)}\, dt + \int_{-\infty}^{0} e^{-t(j\omega-1)}\, dt$$

$$= \frac{-1}{1+j\omega}\, e^{-t(1+j\omega)}\Big|_{0}^{\infty} + \frac{-1}{j\omega-1}\, e^{-t(j\omega-1)}\Big|_{-\infty}^{0}$$

$$= \frac{-1}{1+j\omega}(0-1) + \frac{-1}{j\omega-1}(1-0) = \frac{1}{1+j\omega} + \frac{1}{1-j\omega}$$

$$= \frac{1-j\omega+1+j\omega}{1+\omega^2} = \frac{2}{1+\omega^2}$$

/Ex

Ex $x(t) = \delta(t-2)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-2)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=2}$$

$$= e^{-j2\omega}$$

/Ex

Ex $x(t) = \cos(2t)$

If you try to use $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$, it will not work!

Think back to the CTFS.

We also know that $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

and $x(t) = \frac{1}{2} e^{j2t} + \frac{1}{2} e^{-j2t}$.

what value of $X(j\omega)$ makes this work out?

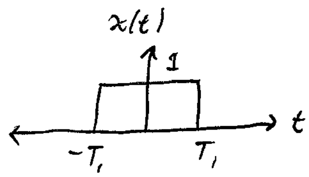
$$X(j\omega) = \pi \delta(\omega - 2) + \pi \delta(\omega + 2)$$

Double-check:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\pi \delta(\omega-2) + \pi \delta(\omega+2)\right] e^{j\omega t} d\omega \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \delta(\omega-2) e^{j\omega t} d\omega + \frac{1}{2} \int_{-\infty}^{\infty} \delta(\omega+2) e^{j\omega t} d\omega \\
&= \frac{1}{2} e^{j\omega t} \Big|_{\omega=2} + \frac{1}{2} e^{j\omega t} \Big|_{\omega=-2} \\
&= \frac{1}{2} e^{j2t} + \frac{1}{2} e^{-j2t} \\
&= \cos(2t)
\end{aligned}$$

/Ex

Ex $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{ow} \end{cases}$

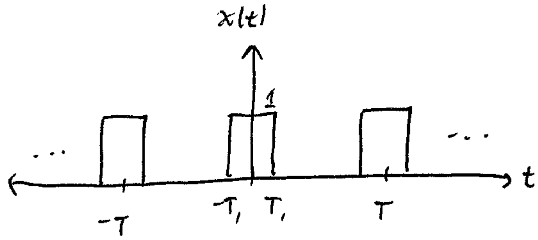


$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
&= \int_{-T_1}^{T_1} (1)\, e^{-j\omega t} dt = \frac{-1}{j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1} \\
&= \frac{-1}{j\omega} \left(e^{-j\omega T_1} - e^{j\omega T_1}\right) = \frac{2j \sin(\omega T_1)}{j\omega} = \frac{2 \sin(\omega T_1)}{\omega}
\end{aligned}$$

This should not be surprising since we got a sinc for a similar function when using the CTFS.

/Ex

Ex



This is a periodic signal, and as just mentiond we found its CTFS:

$$a_k = \frac{\sin\left(\frac{2\pi k T_1}{T}\right)}{\pi k}$$

What is $X(j\omega)$?

We know $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$ but we also know that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega, \quad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} e^{-j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{-\infty}^{\infty} e^{-jt(\omega - \frac{2\pi}{T}k)}\, dt}_{2\pi\delta(\omega - \frac{2\pi}{T}k)}$$ because of the orthogonality

$$= \sum_{k=-\infty}^{\infty} \frac{2\sin\left(\frac{2\pi}{T}T_1 k\right)}{k} \delta\left(\omega - \frac{2\pi}{T}k\right)$$

Double-check:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{2\sin\left(\frac{2\pi T_1}{T}k\right)}{k} \delta\left(\omega - \frac{2\pi}{T}k\right) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{2\sin\left(\frac{2\pi T_1}{T}k\right)}{k} \int_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}k\right) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{2\sin\left(\frac{2\pi T_1}{T}k\right)}{k} e^{j\omega t}\Big|_{\omega = \frac{2\pi}{T}k}$$

$$= \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{2\pi T_1}{T}k\right)}{\pi k} e^{j\frac{2\pi}{T}kt}$$

/Ex

Connection to the CTFS: use a_k's and use δ's to get the correct value of ω $\left(\frac{2\pi}{T}k\right)$.