CTFT & Systems

Recall the output of an LTI system is the convolution of the input & the impulse response:

$$y(t) = x(t) * h(t)$$

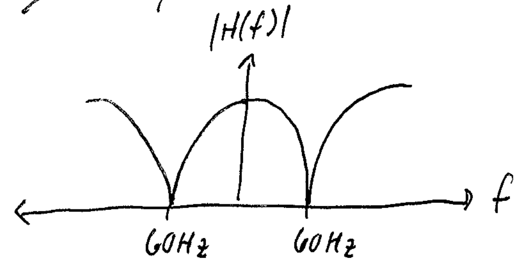
We know from the properties of the CTFT that

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

This means the spectrum of the output is the spectrum of the input, $X(j\omega)$, times the frequency response of the system, $H(j\omega)$.

So, we can use $H(j\omega)$ to shape $Y(j\omega)$.

Ex. There is often a 60 Hz noise created by power supplies (60 Hz is the AC frequency in the US), so we may want to get rid of all 60 Hz signals. To do so, we may create a "filter" with frequency response:



Ex

Before we can start designing frequency responses, though, we have to know how to find them.

Ex: $y(t) = x(t) - \frac{1}{2} y(t)$

$$Y(j\omega) = X(j\omega) - \frac{1}{2} e^{-j\omega} Y(j\omega)$$

$$Y(j\omega)\left(1 + \frac{1}{2} e^{-j\omega}\right) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + \frac{1}{2} e^{-j\omega}}$$

/Ex

Ex $\frac{d}{dt} y(t) + a y(t) = x(t)$

$$j\omega Y(j\omega) + a Y(j\omega) = X(j\omega)$$

$$Y(j\omega)(a + j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega}$$

$h(t) = e^{-at} u(t)$ This was a homework problem!

/Ex

<u>Linear Constant Coefficient Differential Equations</u>

Many systems can be modeled as things of the form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

If we think about it in the frequency domain,

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$Y(j\omega) \sum_{k=0}^{N} a_k (j\omega)^k = X(j\omega) \sum_{k=0}^{M} b_k (j\omega)^k$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Ex: $\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 3x(t)$

$$(j\omega)^2 Y(j\omega) + 6j\omega Y(j\omega) + 5Y(j\omega) = j\omega X(j\omega) + 3X(j\omega)$$

$$(5 + 6j\omega + (j\omega)^2) Y(j\omega) = (3 + j\omega) X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3 + j\omega}{5 + 6j\omega + (j\omega)^2} = \frac{3 + j\omega}{(5+j\omega)(1+j\omega)}$$

$$= \frac{A}{5+j\omega} + \frac{B}{1+j\omega}$$

$A(1+j\omega) + B(5+j\omega) \Rightarrow A + 5B = 3$ $\quad -4A = 3 \quad A = -\frac{3}{4}$

$j\omega(A+B) = 1j\omega \quad B = -A \quad B = \frac{3}{4}$

$$H(j\omega) = \frac{-\frac{3}{4}}{5+j\omega} + \frac{\frac{3}{4}}{1+j\omega}$$

$$h(t) = -\frac{3}{4} e^{-5t} u(t) + \frac{3}{4} e^{-t} u(t)$$

/Ex

Ex : Let $h(t) = e^{-2t} u(t)$ Find $y(t)$

$x(t) = \cos(\pi t)$

$$H(j\omega) = \int_0^\infty e^{-2t} e^{-j\omega t} \, dt = \frac{1}{2+j\omega}$$

$$X(j\omega) = \pi \delta(\omega - \pi) + \pi \delta(\omega + \pi)$$

$$Y(j\omega) = H(j\omega) X(j\omega) = \frac{1}{2+j\omega} \left[\pi \delta(\omega - \pi) + \pi \delta(\omega + \pi) \right]$$

$$= \frac{\pi}{2+j\pi} \delta(\omega - \pi) + \frac{\pi}{2-j\pi} \delta(\omega + \pi)$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2(2+j\pi)} e^{j\pi t} + \frac{1}{2(2-j\pi)} e^{-j\pi t}$$

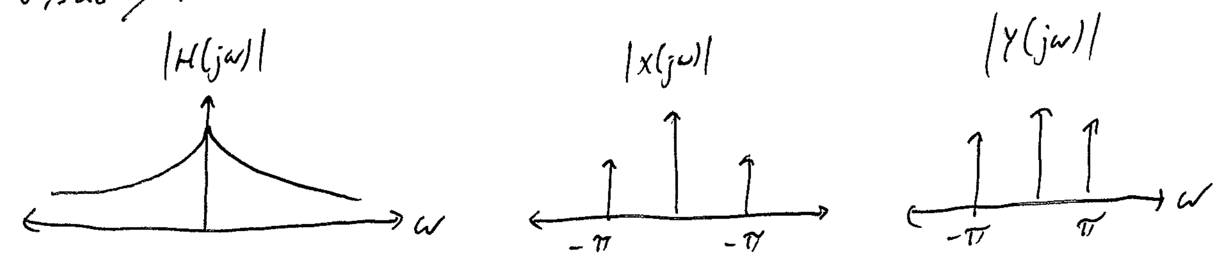
$$= \frac{1}{2} \left[\frac{2-j\pi}{4 - j2\pi + j2\pi + \pi^2} e^{j\pi t} + \frac{2+j\pi}{4 + j2\pi - j2\pi + \pi^2} e^{-j\pi t} \right]$$

$$= \frac{1}{2} \left(\frac{1}{4+\pi^2} \right) \left[2e^{j\pi t} + 2e^{-j\pi t} - j\pi \left(e^{j\pi t} - e^{-j\pi t} \right) \right]$$

$$= \frac{1}{2(4+\pi^2)} \left(4\cos(\pi t) - j\pi \left(2j\sin(\pi t) \right) \right)$$

$$= \frac{2\cos(\pi t)}{4+\pi^2} + \frac{\sin(\pi t)}{4+\pi^2}$$

Visually :



frequency π in $\rightarrow$ frequency π out!

/Ex