LTI Systems: Impulse Response

Lots of systems can be thought of (or approximated as) LTI systems.

Ex: In fMRI, the basic assumption is that the transfer function from stimulus to neural activity is LTI. This is clearly not true, but it works well enough to produce results.

A communications channel (ethernet, fiber optics, wireless) is usually treated as an LTI system.

Ex Circuits in general are LTI.

Anything that can be thought of as having no "long-term memory" (time-invariant) and is linear is an LTI system.

Discrete-Time

Let a system, $T(\cdot)$, be LTI.

This implies that, given $y[n] = T(x[n])$

$$T(x[n-n_0]) = y[n-n_0] \qquad \text{(shifted input} \Rightarrow \text{shifted output)}$$

Also, remember we can write any signal as a linear combination of impulse responses: $x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$

Let's define $h[n] = T(\delta[n])$ as the impulse response of system $T(\cdot)$.

$$\begin{aligned} y[n] &= T(x[n]) \\ &= T\Big(\sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]\Big) \\ &= \sum_{m=-\infty}^{\infty} x[m]\,T(\delta[n-m]) && \text{since } T(\cdot) \text{ is linear} \\ &= \sum_{m=-\infty}^{\infty} x[m]\,h[n-m] && \text{since } T(\cdot) \text{ is time-invariant.} \end{aligned}$$

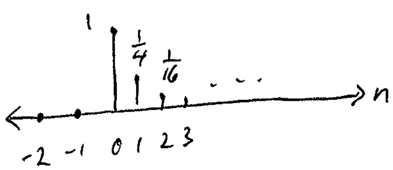
This means that if we know the output to an impulse, we can compute the output to <u>ANY</u> input signal!

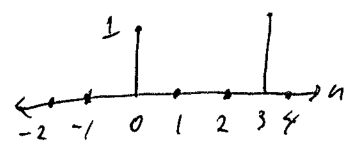
The output is just a bunch of $h[n]$'s scaled and shifted (superposition).

Also, $y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$ is the convolution of $x[n]$ & $h[n]$

$$y[n] = x[n] * h[n]$$
$$= \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m].$$

Ex: Let $h[n] = \left(\frac{1}{4}\right)^n u[n]$ & $x[n] = \delta[n] + \delta[n-3]$





$$y[n] = \sum_{m=-\infty}^{\infty} \left(\delta[m] + \delta[m-3]\right) \left(\tfrac{1}{4}\right)^{n-m} u[n-m]$$
$$= \sum_{m=-\infty}^{\infty} \delta[m] \left(\tfrac{1}{4}\right)^{n-m} u[n-m] + \sum_{m=-\infty}^{\infty} \delta[m-3] \left(\tfrac{1}{4}\right)^{n-m} u[n-m]$$

$$\delta[m] = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} \qquad \delta[m-3] = \begin{cases} 1 & m = 3 \\ 0 & m \neq 3 \end{cases}$$

$$= \left(\tfrac{1}{4}\right)^{n-0} u[n-0] + \left(\tfrac{1}{4}\right)^{n-3} u[n-3]$$

/Ex

Lesson: $h[n] * \delta[n-n_0] = h[n-n_0]$

If you convolve with a delta, just shift the function to be centered at n_0.

More on convolution later.

Continuous-Time

Let a system, $T(\cdot)$, be LTI.

This implies that, given $y(t) = T(x(t))$

$T(x(t-t_0)) = y(t-t_0)$ (shifted input $\Rightarrow$ shifted output)

Also, remember we can write *any* signal as a linear combination of impulse responses: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$

Let's define $h(t) = T(\delta(t))$ as the impulse response.

$$
\begin{aligned}
y(t) &= T(x(t)) \\
&= T\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right) \\
&= \int_{-\infty}^{\infty} x(\tau)\,T(\delta(t-\tau))\,d\tau && \text{since } T(\cdot) \text{ is linear} \\
&= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau && \text{since } T(\cdot) \text{ is time-invariant}
\end{aligned}
$$

This means we can find the output for *any* input just by knowing the impulse response!

Also, $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$ — the convolution of $x(t)$ & $h(t)$

$= x(t) * h(t)$

$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$

Note: This is the same idea as discrete-time. Just replace summations with integrals.

Ex: $h(t) = e^{-2t}u(t)$ & $x(t) = \delta(t) - \frac{1}{2}\delta(t-4)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(\delta(\tau) - \tfrac{1}{2}\delta(\tau-4)\right) e^{-2(t-\tau)}u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau)\, e^{-2(t-\tau)}u(t-\tau)\, d\tau + \int_{-\infty}^{\infty} -\tfrac{1}{2}\delta(\tau-4)\, e^{-2(t-\tau)}u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2(t-0)}u(t-0)\,\delta(\tau)\, d\tau - \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-2(t-4)}u(t-4)\,\delta(\tau-4)\, d\tau$$

$$= e^{-2t}u(t)\underbrace{\int_{-\infty}^{\infty}\delta(\tau)\, d\tau}_{1} - \tfrac{1}{2}e^{-2(t-4)}u(t-4)\underbrace{\int_{-\infty}^{\infty}\delta(\tau-4)\, d\tau}_{1}$$

/Ex $$= e^{-2t}u(t) - \tfrac{1}{2}e^{-2(t-4)}u(t-4)$$

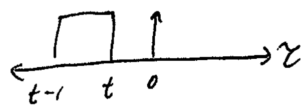
Lesson: $h(t) * \delta(t-t_0) = h(t-t_0)$

Similar to discrete-time: if you convolve with a delta, you just shift the function to be centered at t_0.

Finding the impulse response

make the input an impulse ($\delta(t)$ or $\delta[n]$) and the output is then the impulse response ($h(t)$ or $h[n]$)

Ex: $y(t) = \int_{t-1}^{t} x(\tau)\,d\tau$



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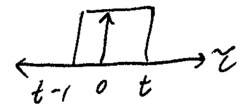

$$h(t) = \int_{t-1}^{t} \delta(\tau)\,d\tau$$

$$= \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o w} \end{cases}$$

$h(t) = \mathcal{U}(t) - \mathcal{U}(t-1)$

/Ex

Ex $y[n] = x[n] - \frac{1}{2}\cancel{x}[n-1]$

$h[n] = \delta[n] - \frac{1}{2}h[n-1]$

$h[n] = 0$ for $n<0$ since $\delta[n] = 0$ for $n<0$

$h[0] = \delta[0] - \frac{1}{2}h[-1] = 1 - \frac{1}{2}(0) = 1$

$h[1] = \delta[1] - \frac{1}{2}h[0] = 0 - \frac{1}{2}(1) = -\frac{1}{2}$

$h[2] = \delta[2] - \frac{1}{2}h[1] = 0 - \frac{1}{2}(-\frac{1}{2}) = \frac{1}{4}$

$\vdots$

$h[n] = (-\frac{1}{2})^n\,\mathcal{U}[n]$

/Ex