Fourier Series Properties

Linearity

If $x(t) \xleftrightarrow{FS} a_k$ with the same period, T

$y(t) \xleftrightarrow{FS} b_k$

Then $z(t) = \alpha x(t) + \beta y(t) \xleftrightarrow{FS} \alpha a_k + \beta b_k$

why: Let $z(t) \xleftrightarrow{FS} c_k$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\frac{2\pi}{T}t}\,dt$$

$$= \frac{1}{T}\int_T \left(\alpha x(t) + \beta y(t)\right) e^{-jk\frac{2\pi}{T}t}\,dt$$

$$= \frac{1}{T}\alpha\int_T x(t) e^{-jk\frac{2\pi}{T}t}\,dt + \beta\frac{1}{T}\int_T y(t) e^{-jk\frac{2\pi}{T}t}\,dt$$

$$= \alpha a_k + \beta b_k$$

If $x[n] \xleftrightarrow{FS} a_k$ with the same period, N

$y[n] \xleftrightarrow{FS} b_k$

Then $z[n] = \alpha x[n] + \beta y[n] \xleftrightarrow{FS} \alpha a_k + \beta b_k$

why: Let $z[n] \xleftrightarrow{FS} c_k$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} z[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left(\alpha x[n] + \beta y[n]\right) e^{-jk\frac{2\pi}{N}n}$$

$$= \alpha\frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} + \beta\frac{1}{N}\sum_{n=0}^{N-1} y[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \alpha a_k + \beta b_k$$

Time Shifting

If $x(t) \xleftrightarrow{FS} a_k$ and $y(t) = x(t-t_0)$

Then $y(t) \xleftrightarrow{FS} e^{-jk\frac{2\pi}{T}t_0} a_k$

why: Let $y(t) \xleftrightarrow{FS} b_k$

$$b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\frac{2\pi}{T}t}\, dt \qquad \text{Let } u = t-t_0 \quad du = dt$$

$$= \frac{1}{T}\int_T x(u)\, e^{-jk\frac{2\pi}{T}(u+t_0)}\, du$$

$$= e^{-jk\frac{2\pi}{T}t_0}\, \frac{1}{T}\int_T x(u)\, e^{-jk\frac{2\pi}{T}u}\, du$$

$$= e^{-jk\frac{2\pi}{T}t_0}\, a_k$$

If $x[n] \xleftrightarrow{FS} a_k$ and $y[n] = x[n-n_0]$

Then $y(t) \xleftrightarrow{FS} e^{-jk\frac{2\pi}{N}n_0} a_k$

why: Let $y[n] \xleftrightarrow{FS} b_k$

$$b_k = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]\, e^{-jk\frac{2\pi}{N}n} \qquad \text{Let } \ell = n-n_0$$

$$= \frac{1}{N}\sum_{\ell=-n_0}^{N-n_0-1} x[\ell]\, e^{-jk\frac{2\pi}{N}(\ell+n_0)}$$

$$= e^{-jk\frac{2\pi}{N}n_0}\, \frac{1}{N}\sum_{\ell=-n_0}^{N-n_0-1} x[\ell]\, e^{-jk\frac{2\pi}{N}\ell}$$

$$= e^{-jk\frac{2\pi}{N}n_0}\, a_k$$

Time Reversal

If $x(t) \xleftrightarrow{FS} a_k$ and $y(t) = x(-t)$

Then $y(t) \xleftrightarrow{FS} a_{-k}$

If $x[n] \xleftrightarrow{FS} a_k$ and $y[n] = x[-n]$

Then $y(t) \xleftrightarrow{FS} a_{-k}$

~~[illegible]~~

Multiplication

If $x(t) \xleftrightarrow{FS} a_k$, and they have the same period, T

$y(t) \xleftrightarrow{FS} b_k$

Then $x(t)y(t) \xleftrightarrow{FS} \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$ Note: This is $a_k * b_k$

why: Let $x(t)y(t) \xleftrightarrow{FS} c_k$

$$c_k = \frac{1}{T}\int_T x(t)y(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\int_T \left[\sum_{n=-\infty}^{\infty} a_n e^{+jn\frac{2\pi}{T}t}\right]\left[\sum_{m=-\infty}^{\infty} b_m e^{+jm\frac{2\pi}{T}t}\right] e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n b_m \int_T e^{+j\frac{2\pi}{T}(n+m-k)t}\, dt$$

Remember: $\int_T e^{+j\frac{2\pi}{T}(n+m-k)t}\, dt = \begin{cases} T & n+m-k=0 \\ 0 & ow \end{cases}$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n b_m\, T\,\delta(m+(n-k))$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} a_n b_{k-n}\, T = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

If $x[n] \xleftrightarrow{FS} a_k$ they both have the same period, N.
$y[n] \xleftrightarrow{FS} b_k$

Then $x[n]\,y[n] \xleftrightarrow{FS} \sum_{\ell=\langle N\rangle} a_\ell b_{k-\ell}$

This is the ~~circular~~ convolution of a_k & b_k over a period.

Why: Let $x[n]\,y[n] \xleftrightarrow{FS} c_k$

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\,y[n]\,e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\left[\sum_{\ell=\langle N\rangle} a_\ell e^{j\ell\frac{2\pi}{N}n}\right]\left[\sum_{m=\langle N\rangle} b_m e^{jm\frac{2\pi}{N}n}\right]e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{\ell=\langle N\rangle}\sum_{m=\langle N\rangle} a_\ell b_m \sum_{n=\langle N\rangle} e^{j\frac{2\pi}{N}(\ell+m-k)n}$$

$$= \frac{1}{N}\sum_{\ell=\langle N\rangle}\sum_{m=\langle N\rangle} a_\ell b_m\, N\,\delta(m+(\ell-k))$$

$$= \frac{1}{N}\sum_{\ell=\langle N\rangle} a_\ell b_{k-\ell} N = \sum_{\ell=\langle N\rangle} a_\ell b_{k-\ell}$$

Take-home message:

Multiplication in time $\Rightarrow$ convolution in frequency ☆

Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof: $\frac{1}{T}\int_T |x(t)|^2\,dt = \frac{1}{T}\int_T x(t)\,x^*(t)\,dt$

$$= \frac{1}{T}\int_T \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}\right]\left[\sum_{\ell=-\infty}^{\infty} a_\ell e^{j\ell\frac{2\pi}{T}t}\right]^* dt$$

$$= \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k a_\ell^* e^{jk\frac{2\pi}{T}t} e^{-j\ell\frac{2\pi}{T}t}\,dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \int_T e^{j\frac{2\pi}{T}(k-\ell)t}\,dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k a_\ell^*\, T\,\delta(k-\ell) = \sum_{k=-\infty}^{\infty} a_k a_k^* = \sum_{k=-\infty}^{\infty} |a_k|^2$$

/Proof

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Proof: $$\frac{1}{N}\sum_{n=\langle N\rangle}\left[\sum_{\ell=\langle N\rangle} a_\ell e^{j\ell\frac{2\pi}{N}n}\right]\left[\sum_{m=\langle N\rangle} a_m e^{jm\frac{2\pi}{N}n}\right]^*$$

$$= \frac{1}{N}\sum_{\ell=\langle N\rangle}\sum_{m=\langle N\rangle} a_\ell a_m^* \sum_{n=\langle N\rangle} e^{j\frac{2\pi}{N}(\ell-m)n}$$

$$= \frac{1}{N}\sum_{\ell=\langle N\rangle}\sum_{m=\langle N\rangle} a_\ell a_m^* N\delta(\ell-m)$$

$$= \frac{1}{N}\sum_{\ell=\langle N\rangle} N a_\ell a_\ell^* = \sum_{\ell=\langle N\rangle} |a_\ell|^2$$

/Proof

Take-home message:

The energy in the time-domain ~~sig~~ signal is the same as the energy in the frequency domain.