CTFT Properties

Linearity

If $x(t) \xleftrightarrow{CTFT} X(j\omega)$ & $y(t) \xleftrightarrow{CTFT} Y(j\omega)$

Then $a x(t) + b y(t) \xleftrightarrow{CTFT} a X(j\omega) + b Y(j\omega)$

Proof: Let $z(t) = a x(t) + b y(t)$

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} (a x(t) + b y(t)) e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

/Proof $$= a X(j\omega) + b Y(j\omega)$$

This is nice since it lets you use CTFT pairs you already know to find another CTFT pair.

It also means if you add a new signal (interference, noise, etc) to an existing signal, you just add the spectra and don't destroy the original signal.

Time Shifting

If $x(t) \xleftrightarrow{CTFT} X(j\omega)$

Then $x(t-t_0) \xleftrightarrow{CTFT} e^{-j\omega t_0} X(j\omega)$

Proof: Let $y(t) = x(t-t_0)$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

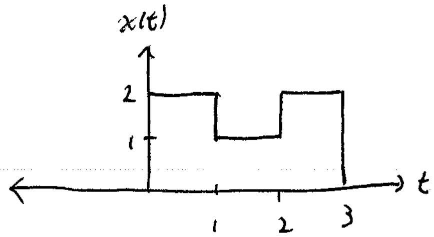
Let $U = (t-t_0)$, $dU = dt$

$$= \int_{-\infty}^{\infty} x(U) e^{-j\omega(U+t_0)} dU$$

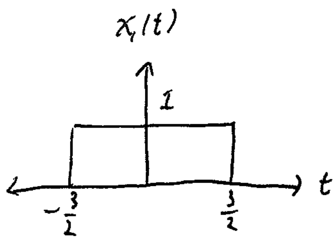
$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(U) e^{-j\omega U} dU$$

/Proof $$= e^{-j\omega t_0} X(j\omega)$$

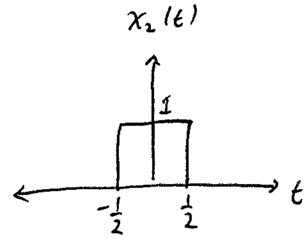
Ex: Find $X(j\omega)$



Let



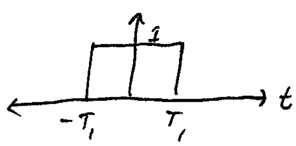
and



So, ~~with~~ $x(t) = 2x_1(t-\frac{3}{2}) - x_2(t-\frac{3}{2})$

$$X(j\omega) = 2e^{-j\frac{3}{2}\omega}X_1(j\omega) - e^{-j\frac{3}{2}\omega}X_2(j\omega)$$

we also know that



$$\xleftrightarrow{CTFT} \frac{2\sin(\omega T_1)}{\omega}$$

So, $X_1(j\omega) = \frac{2\sin(\frac{3}{2}\omega)}{\omega}$ & $X_2(j\omega) = \frac{2\sin(\frac{\omega}{2})}{\omega}$

$$X(j\omega) = \frac{4e^{-j\frac{3}{2}\omega}\sin(\frac{3}{2}\omega)}{\omega} - \frac{2e^{-j\frac{3}{2}\omega}\sin(\frac{\omega}{2})}{\omega}$$

/Ex

<u>Conjugation</u>

If $x(t) \xleftrightarrow{CTFT} X(j\omega)$

Then $x^*(t) \xleftrightarrow{CTFT} X^*(-j\omega)$

You should be able to show this in only a few lines.

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Again, the energy in the time domain is the same as the energy in the frequency domain.

Convolution

If $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$, $h(t) \overset{CTFT}{\longleftrightarrow} H(j\omega)$, and $y(t) = h(t) * x(t)$

Then $Y(j\omega) = H(j\omega) X(j\omega)$

convolution in time $\Longleftrightarrow$ multiplication in frequency (just like CTFS)

So, for a system with impulse response $h(t)$, we refer to $H(j\omega)$ as the frequency response of the system.

This information captures the important information of the system – ie, how it responds to each frequency

Duality

Taking a look at the CTFS formulas, one observes a symmetry

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \qquad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

Let's see what happens if we "flip x & X"

Assume we have a known CTFT pair: $x(t) \overset{CTFT}{\longleftrightarrow} X(\omega)$

What is the CTFT of $X(t)$? the inverse CTFT of $x(\omega)$?

$$\mathfrak{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} \, dt$$

$$= 2\pi x(-\omega)$$

$$\mathfrak{F}^{-1}\{x(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} X(-t)$$

So if we know $x(t) \xleftrightarrow{CTFS} X(\omega)$

then we know $X(t) \xleftrightarrow{CTFS} 2\pi\, x(-\omega)$ and

$\frac{1}{2\pi} X(t) \xleftrightarrow{CTFS} x(\omega)$ (these are equivalent)

Ex: Find the CTFT of $\frac{1}{2+j3t}$

If you look at the table on PG 329, you find the pair

$$e^{-at}\mathcal{U}(t) \xleftrightarrow{CTFT} \frac{1}{a+j\omega}$$

We have $X(t) = \frac{1}{2+j3t} = \frac{1}{3}\,\frac{1}{\frac{2}{3}+jt}$

$X(t) \xleftrightarrow{CTFS} 2\pi\, x(-\omega)$ where $x(\omega) = \frac{1}{3} e^{-\frac{2}{3}\omega}\mathcal{U}(\omega)$

/Ex So, $\mathcal{F}\left\{\frac{1}{2+j3t}\right\} = \frac{2\pi}{3} e^{\frac{2}{3}\omega}\mathcal{U}(-\omega)$

It is often ~~far~~ easier to compute one way vs the other.

Ex: $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$

We already found $X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$

Now suppose you were given $x(t) = \frac{2\sin(Wt)}{t}$

Good luck computing $X(j\omega)$ directly.

But using duality it's easy: $X(j\omega) = \begin{cases} 2\pi & |\omega| < W \\ 0 & \text{ow} \end{cases}$