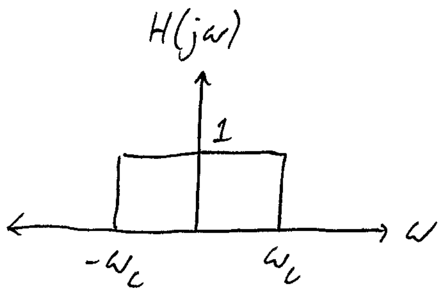
Common Applications of CTFT

Filtering

Suppose you have a band of frequencies you want to isolate/reject.

Ex: Ideal LPF (low-pass filter)



This filter only lets frequencies between $\pm\omega_c$ through.

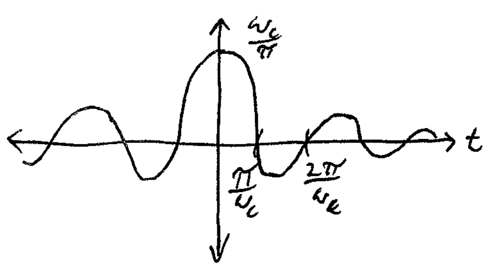
Let's look at the impulse response.

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} (1)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\frac{1}{jt}e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\pi t}\frac{1}{2j}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)$$

$$= \frac{\sin(\omega_c t)}{\pi t}$$



Problems:

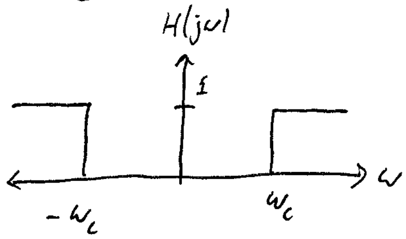
1) Is this system causal?
   NO! $h(t) \neq 0$ for $t<0$

2) Is the impulse response finite duration?
   NO!

So, we could never build a system with this impulse response, but it will be a tool we will use in principle.

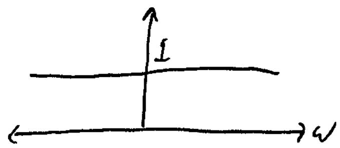
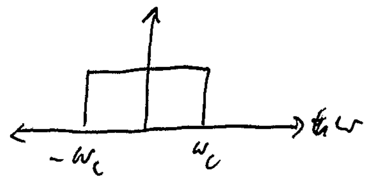
Ex Real LPF's will approximate this frequency response.

Ex Ideal High-Pass Filter (HPF)



This filter only lets frequencies above $\pm\omega_c$ through.

Note:

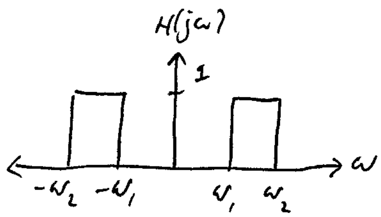
$H(j\omega) =$  $-$ 

$$\text{So, } h(t) = \mathcal{F}^{-1}\{1\} - \frac{\sin(\omega_c t)}{\pi t}$$

$$= \delta(t) - \frac{\sin(\omega_c t)}{\pi t}$$

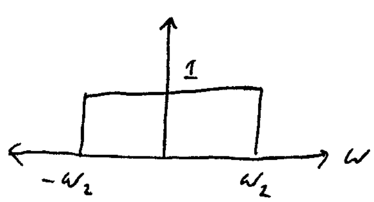
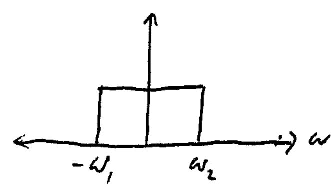
/Ex This filter is also unrealizable.

Ex Ideal Band-Pass Filter (BPF)



This filter only lets in frequencies between ω_1 & ω_2

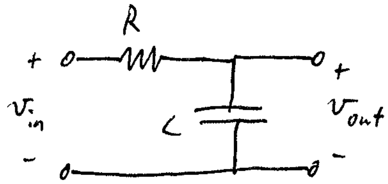
Note:

$H(j\omega) =$  $-$ 

$$\text{So, } h(t) = \frac{\sin(\omega_2 t)}{\pi t} - \frac{\sin(\omega_1 t)}{\pi t}$$

/Ex This filter is also unrealizable.

Ex Simple LPF : RC circuit



Let's try to find $h(t)$ of the system.

Start with $v_{in}(t) = u(t)$

Recall from circuits that $v_{out}(t) = \left(1 - e^{-\frac{t}{RC}}\right) u(t)$

Now we can find $h(t)$ by taking a derivative.

$$h(t) = \frac{d}{dt}\left(1 - e^{-\frac{t}{RC}}\right) u(t)$$

$$= \left(\frac{1}{RC} e^{-\frac{t}{RC}}\right) u(t) + \left(1 - e^{-\frac{t}{RC}}\right) \delta(t)$$

$$= \left(\frac{1}{RC} e^{-\frac{t}{RC}}\right) u(t) + \left(1 - e^{\frac{-(0)}{RC}}\right) \delta(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

Let's look at $H(j\omega)$ now

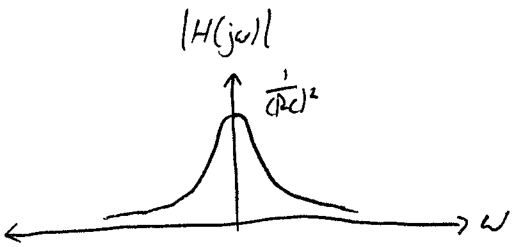
$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-\frac{t}{RC}} e^{-j\omega t} dt = \frac{1}{RC} \int_{0}^{\infty} e^{-t\left(\frac{1}{RC} + j\omega\right)} dt$$

$$= \frac{1}{RC} \left(\frac{-1}{RC + j\omega}\right) e^{-t\left(\frac{1}{RC} + j\omega\right)} \Big|_{0}^{\infty} = \frac{1}{RC} \frac{1}{RC + j\omega}$$

What is $|H(j\omega)|$, though?

$$|H(j\omega)| = \frac{1}{RC} \left| \frac{1}{RC + j\omega} \right| = \frac{1}{RC} \frac{1}{\sqrt{(RC)^2 + \omega^2}}$$



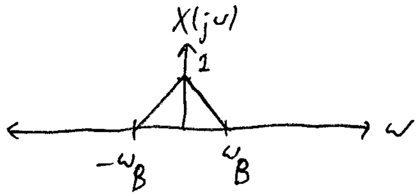
So, this circuit still picks up mostly low frequencies, and adjusting the value of RC will make $|H(j\omega)|$ more broad/narrow.

/Ex

AM Modulation

Let's say we have a signal of interest, $x(t)$, (voice, music, data, etc)

and let's say it is band-limited, $X(j\omega) = 0$ for $|\omega| > \omega_B$



AM-DSB

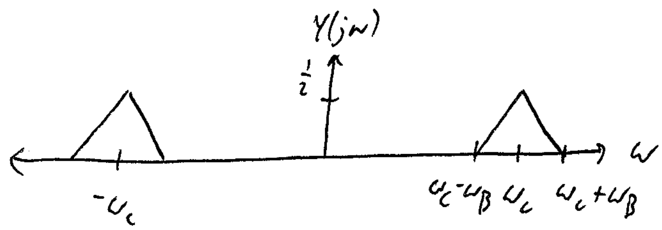
Then let's multiply $x(t)$ by $\cos(\omega_c t)$ & $\omega_c \gg \omega_B$

$$y(t) = x(t)\cos(\omega_c t)$$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * \mathcal{F}\{\cos(\omega_c t)\}$$

$$= \frac{1}{2\pi} X(j\omega) * \left(\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)\right)$$

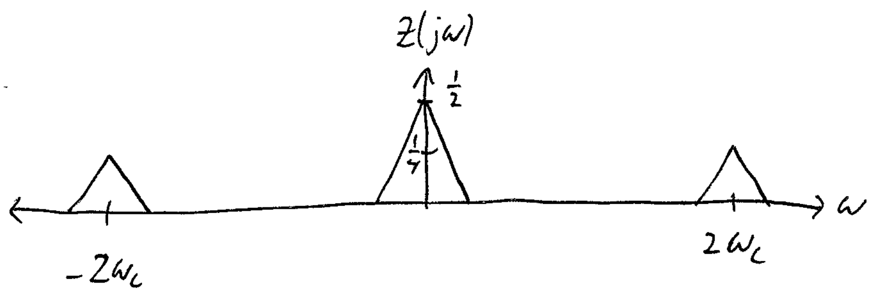
$$= \frac{1}{2} X(j(\omega-\omega_c)) + \frac{1}{2} X(j(\omega+\omega_c))$$



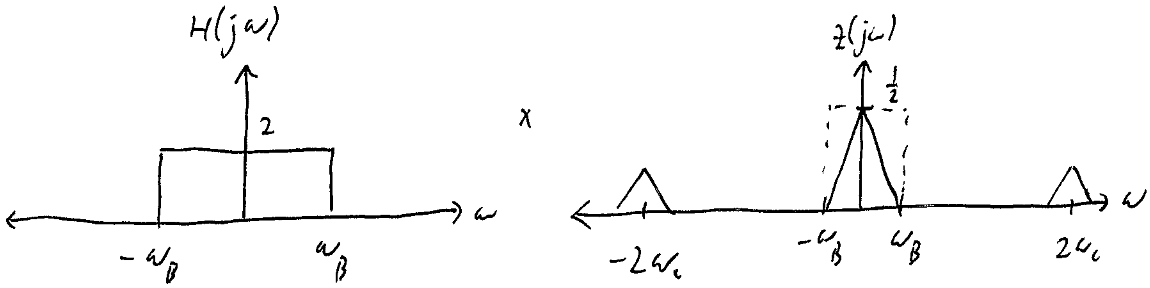
Let's try to recover $x(t)$ back again.

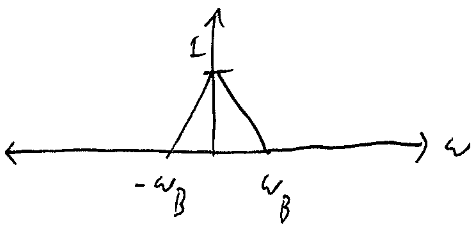
Let $z(t) = y(t)\cos(\omega_c t)$

$$Z(j\omega) = \frac{1}{2} Y(j(\omega-\omega_c)) + \frac{1}{2} Y(j(\omega+\omega_c))$$



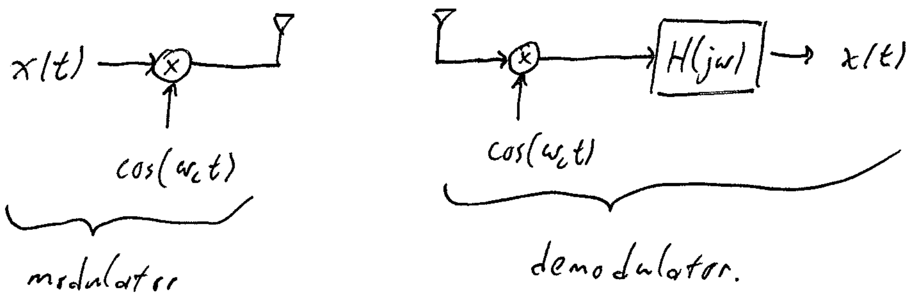
So, now if we LPF $z(t)$, we should get back $x(t)$



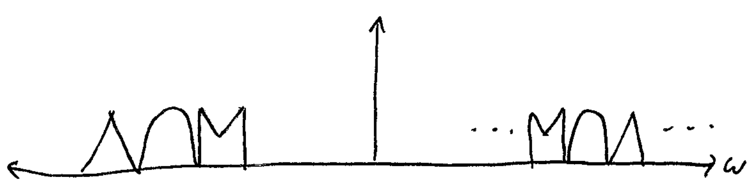


$\rightarrow$ $x(t)$

This is known as AM - DSB (Double Side-Band)



Note that we can have multiple modulated signals without interfering. This is how you can have multiple radio stations.



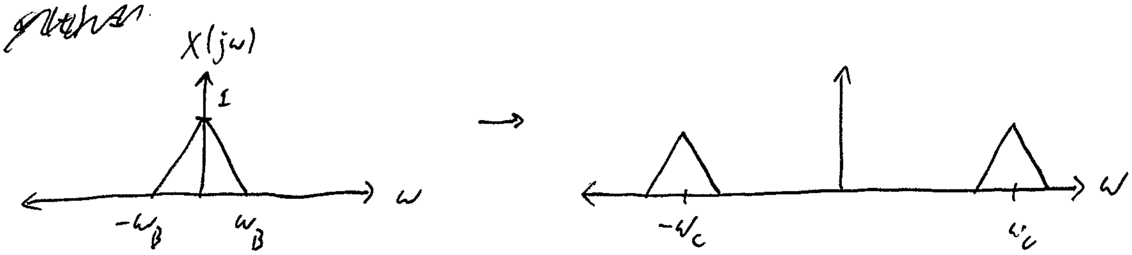
AM-SSB

If you notice in the previous section, we needed to use $2\omega_B$ of the spectrum to represent the modulated signal. However, given that $x(t)$ is real-valued, we really only need to use ω_B.

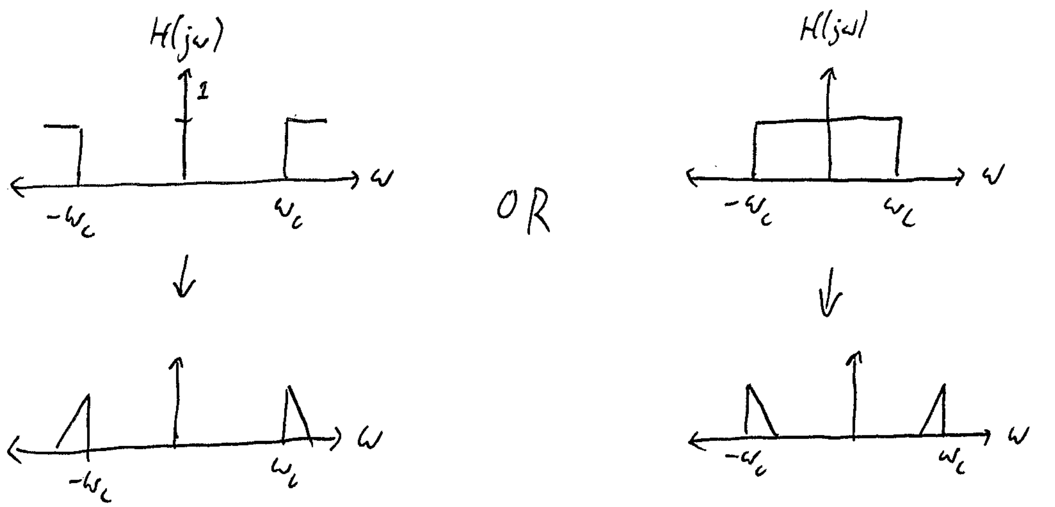
Ex If we see $e^{j\frac{2\pi}{5}t}$, we know we really had $\mathrm{Re}\{e^{j\frac{2\pi}{5}t}\} = \cos(\frac{2\pi}{5}t)$

/Ex This will be clear pictorially.

We will still start by multiplying by $\cos(\omega_c t)$.

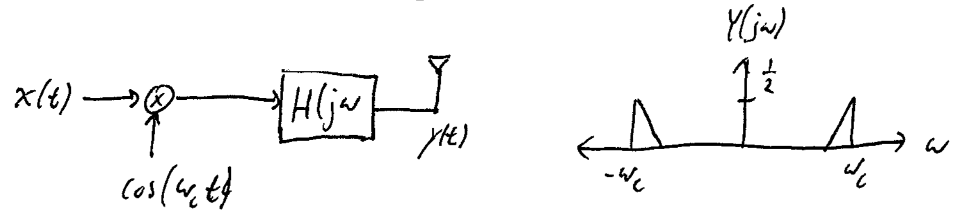


Now we're going to throw away half of the spectrum by filtering. We can either keep the upper half ($|\omega| > \omega_c$) or the lower half ($|\omega| < \omega_c$)

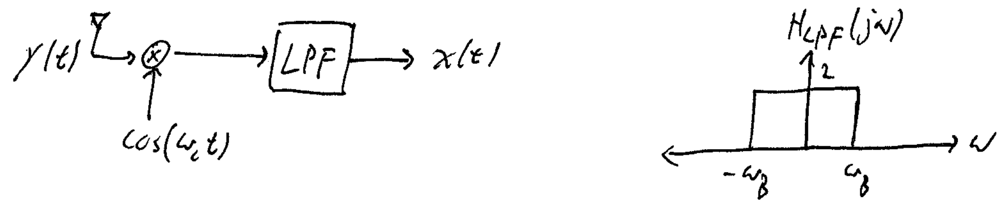


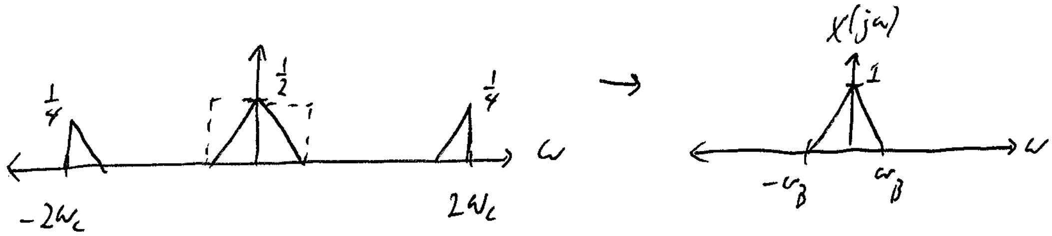
For purposes of demonstration, let's keep the lower side-band.

So, for the modulator, we get:



Now for the demodulator, multiply by $\cos(\omega_c t)$ & LPF.





So, we used half as much space in the spectrum (at the transmitter), but could still recover $x(t)$ perfectly.

It did complicate our transmitter/modulator, but not the receive/demodulator.

Note: Both the DSB & SSB receivers needed to be synchronized with the transmitter carrier.

$$\cos(\omega_c t) \quad \& \quad \cos(\omega_c t)$$

$$\text{NOT} \quad \cos(\omega_c t) \quad \& \quad \cos(\omega_c t + \theta)$$

In practice, radios have circuits called phase-locked loops (PLL's) to estimate the phase offset.

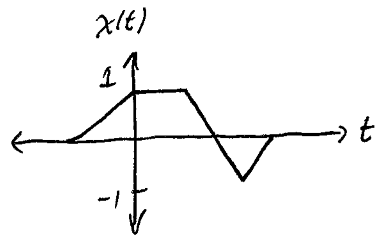
Another approach is to use an asynchronous demodulator.

Asynchronous Demodulation

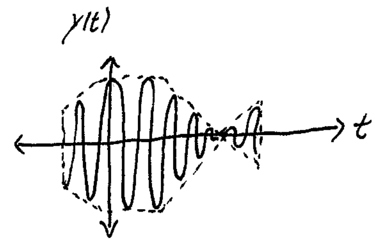
There is a slight modification to the modulator to make the demodulator simple.

$y(t) = (x(t) + A)\cos(\omega_c t)$ where $x(t) + A \geq 0$ for all $t \in \mathbb{R}$

Ex:

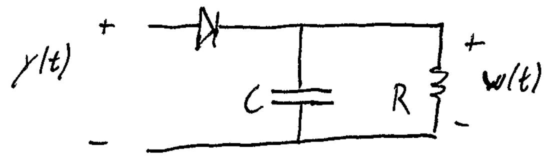


Let $A=1$ →



~~Ex~~

So, now if we can recover the envelope (the dashed lines), we can find $x(t)$.



$w(t) \approx x(t)$

Ex

→

Modulator:

$y(t) = (x(t) + A)\cos(\omega_c t)$

Demodulator: