Impulse Response: System Properties

You can determine the properties of an LTI system based only on its implse response.

This should not be surprising since we already established that we can compute the output to any input using only the impulse response.

This is useful because you can measure/estimate the impulse response to a system without knowing exactly what that system does, providing insight into how the system works

Causality

For a system to be ~~stable~~, causal,

$h[n] = 0$ for $n < 0$ $\qquad$ $h(t) = 0$ for $t < 0$

why: $y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

$$= \sum_{k=-\infty}^{-1} h[k]\, x[n-k] + \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

If any $h[k]$, $k < 0$ is non-zero, $h[-k_0]$ $\quad k_0 > 0$

$y[n]$ depends on $x[n+k_0]$, which is a future value!

Same thing for continuous-time using an integral:

$$y(t) = \int_{-\infty}^{0} h(\tau)\, x(t-\tau)\, d\tau + \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let $h(\tau)$ be non-zero for $-\tau_0$ ~~and~~, $\tau_0 > 0$

$y[n]$ depends on $x(t+\tau_0)$, future value.

Memoryless

For a system to be memoryless,

$h[n] = C\delta[n]$ $\qquad$ $h(t) = C\delta(t)$ $\quad$ where C is a constant.

Why: For a system to be memoryless, $y[n] = Cx[n]$ or $y(t) = Cx(t)$

If $h[n] = C\delta[n]$, $y[n] = x[n] * C\delta[n] = Cx[n]$

If $h(t) = C\delta(t)$, $y(t) = x(t) * C\delta(t) = Cx(t)$

Stability

For a system to be BIBO stable,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \qquad \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Why: Given a bounded input: $|x[n]| < B$ or $|x(t)| < B$

A bounded output implies $|y[n]| < C$ or $|y(t)| < C$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right|$$
$$\leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$
$$\leq \sum_{k=-\infty}^{\infty} |h[k]|\,B$$

So, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, $C = B\sum_{k=-\infty}^{\infty} |h[k]|$ and stability

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau\right|$$
$$\leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau$$
$$\leq \int_{-\infty}^{\infty} |h(\tau)|\,B\,d\tau$$

So, if $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$, $C = B\int_{-\infty}^{\infty} |h(t)|\,dt$ and stability

Ex: $h(t) = u(t+1) - u(t-1)$

It is non-causal $\qquad h(t) = 1$ for $t = -\frac{1}{2}$

It is not memoryless. $\qquad h(t) \neq c\delta(t)$

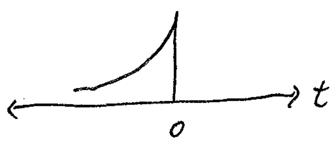
It is stable

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-1}^{1} |(1)|\, dt = 2 < \infty$$

/Ex

Ex $\quad h(t) = 2^t u(-t)$

It is non-causal $\qquad h(t) \neq 0$ for all $t<0$



It is not memoryless $\qquad h(t) \neq c\delta(t)$

It is stable

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{0} 2^t\, dt = \int_0^{\infty} \left(\tfrac{1}{2}\right)^u du$$

$u = -t \quad du = -dt$

$$= \int_0^{\infty} e^{u \ln \frac{1}{2}}\, du = \frac{1}{\ln \frac{1}{2}} e^{u \ln \frac{1}{2}} \Big|_0^{\infty} = \frac{1}{\ln(\frac{1}{2})}(0-1)$$

$$= \frac{1}{\ln 2} < \infty$$

/Ex

Ex $h[n] = u[n] - u[n-1]$

The system is causal : $h[n] = 0$ for $n<0$

The system is memoryless : $h[n] = \delta[n]$!

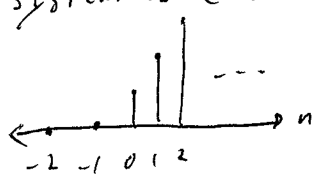
The system is stable :

$$\left|\sum_{n=-\infty}^{\infty} h[n]\right| = \sum_{n=-\infty}^{\infty} |\delta[n]| = 1 < \infty$$

/Ex

Ex $h[n] = 3^n u[n]$

The system is causal : $h[n] = 0$ for $n<0$



The system is not memoryless : $h[n] \neq C\delta[n]$

The system is unstable :

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 3^n = \infty$$

/Ex