Convolution

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \qquad x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Properties

1) Commutative : $x[n] * h[n] = h[n] * x[n]$

$x(t) * h(t) = h(t) * x(t)$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \qquad \text{Let } m = n-k, \; k = n-m$$

$$= \sum_{m=-\infty}^{\infty} x[n-m]\, h[m] = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \qquad \text{Let } u = (t-\tau) \quad d\tau = -du$$

$$= \int_{\infty}^{-\infty} x(t-u)\, h(u)\, (-du)$$

$$= \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du = h(t) * x(t)$$

2) Distributive : $x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$

$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$

$$x[n] * \left(h_1[n] + h_2[n]\right) = \sum_{k=-\infty}^{\infty} x[k] \left(h_1[n-k] + h_2[n-k]\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]\, h_2[n-k]$$

$$= x[k] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * \left(h_1(t) + h_2(t)\right) = \int_{-\infty}^{\infty} x(\tau) \left(h_1(t-\tau) + h_2(t-\tau)\right) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau$$

$$= x(t) * h_1(t-\tau) + x(t) * h_2(t-\tau)$$

3) Associative: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

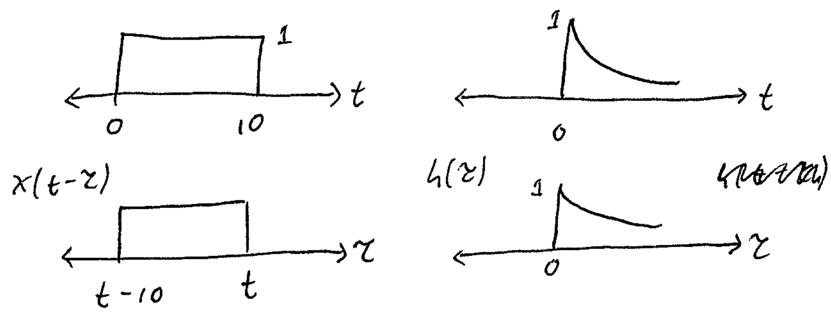
$$x(t) * (h_1(t) * h_2(t)) = (x_1(t) * h_1(t)) * h_2(t)$$

This proof will be a homework problem.

How to compute a convolution result

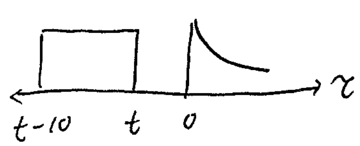
1) Draw the two signals.
2) Flip and shift one of them
3) Determine different cases
4) Compute the integral for each case.
5) Collect the different output cases into one (piecewise) output.

Ex: $x(t) = u(t) - u(t-10)$, $h(t) = e^{-t}u(t)$



$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

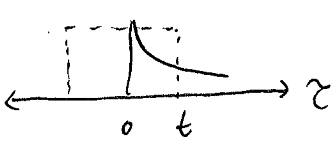
Case 1: $t < 0$



$h(\tau)\,x(t-\tau) = 0$ in this range

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} (0)\, d\tau = 0$$

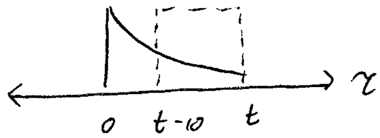
Case 2: $0 \le t < 10$



$$y(t) = \int_0^t e^{-\tau} (1)\, d\tau = -e^{-\tau}\Big|_0^t$$

$$= 1 - e^{-\tau}$$

Case 3: $t \geq 10$



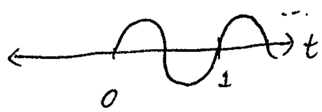
$$y(t) = \int_{t-10}^{t} e^{-\tau}(1)\,d\tau$$

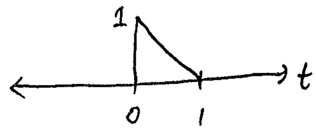
$$= -e^{-\tau}\Big|_{t-10}^{t} = e^{-(t-10)} - e^{-t}$$

$$y(t) = \begin{cases} 1 - e^{-\tau} & 0 \leq t < 10 \\ e^{-(t-10)} - e^{-t} & t \geq 10 \\ 0 & \text{ow} \end{cases}$$

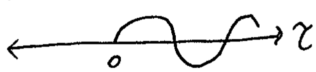
Ex

Ex $\quad x(t) = \sin(2\pi t)\,u(t) \qquad h(t) = (1-t)(u(t) - u(t-1))$

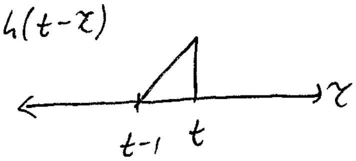




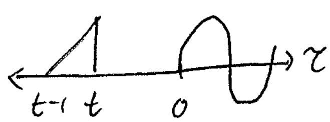
$x(\tau)$



$h(t-\tau)$



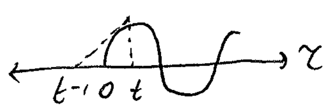
Case 1: $t<0$



$$h(t-\tau)\,x(\tau) = 0$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau = 0$$

Case 2: $0 \leq t < 1$



$$y(t) = \int_{0}^{t} (1-(t-\tau))\sin(2\pi\tau)\,d\tau$$

$$= (1-t)\int_0^t \sin(2\pi\tau)\,d\tau + \int_0^t \tau\sin(2\pi\tau)\,d\tau$$

$u = \tau \qquad dv = \sin(2\pi\tau)\,d\tau$

$du = d\tau \qquad v = \frac{-1}{2\pi}\cos(2\pi\tau)$

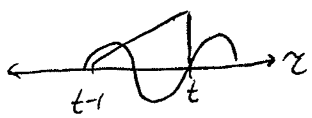
$$y(t) = (1-t)\left(\frac{-1}{2\pi}\right)\cos(2\pi\tau)\Big|_0^t + \left[\frac{-\tau}{2\pi}\cos(2\pi\tau)\Big|_0^t + \frac{1}{2\pi}\int_0^t \cos(2\pi\tau)\,d\tau\right]$$

$$= (1-t)\left(\frac{-1}{2\pi}\right)(\cos(2\pi t) - 1) + \left[\frac{-t}{2\pi}\cos(2\pi t) + \frac{1}{4\pi^2}\sin(2\pi\tau)\Big|_0^t\right]$$

$$= \frac{1-t}{2\pi}(1-\cos(2\pi t)) - \frac{t}{2\pi}\cos(2\pi t) + \frac{1}{4\pi^2}\sin(2\pi t)$$

$$= \frac{1-t}{2\pi} - \frac{1}{2\pi}\cos(2\pi t) + \frac{1}{4\pi^2}\sin(2\pi t)$$

Case 3: $t \geq 1$



$$y(t) = \int_{t-1}^{t} (1-t+\tau)\sin(2\pi\tau)\,d\tau$$

$$= (1-t)\int_{t-1}^{t} \sin(2\pi\tau)\,d\tau + \int_{t-1}^{t} \tau \sin(2\pi\tau)\,d\tau$$

$$y(t) = (1-t)\left(\frac{-1}{2\pi}\right)\cos(2\pi\tau)\Big|_{t-1}^{t} + \left[\frac{-\tau}{2\pi}\cos(2\pi\tau)\Big|_{t-1}^{t} + \frac{1}{2\pi}\int_{t-1}^{t}\cos(2\pi\tau)\,d\tau\right]$$

$$= (1-t)\left(\frac{-1}{2\pi}\right)\left(\cos(2\pi t) - \cos(2\pi(t-1))\right)$$

$$+ \left[\frac{-t}{2\pi}\cos(2\pi t) + \frac{(t-1)}{2\pi}\cos(2\pi(t-1)) + \frac{1}{4\pi^2}\sin(2\pi\tau)\Big|_{t-1}^{t}\right]$$

$$= \left(\frac{t-1}{2\pi}\right)\left(\cos(2\pi t) - \cos(2\pi t - 2\pi)\right)$$

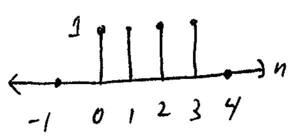
$$+ \left[\frac{-t}{2\pi}\cos(2\pi t) + \frac{(t-1)}{2\pi}\cos(2\pi t - 2\pi) + \frac{1}{4\pi^2}\left(\sin(2\pi t) - \sin(2\pi(t-1))\right)\right]$$

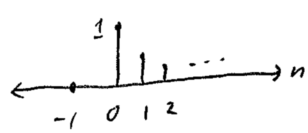
$$= 0 + \frac{\cos(2\pi t)}{2\pi}(-t+t-1) + 0 = \frac{-\cos(2\pi t)}{2\pi}$$

$$y(t) = \begin{cases} \frac{1-t}{2\pi} - \frac{1}{2\pi}\cos(2\pi t) + \frac{1}{4\pi^2}\sin(2\pi t) & 0 \leq t < 1 \\ -\frac{1}{2\pi}\cos(2\pi t) & t \geq 1 \\ 0 & \text{ow} \end{cases}$$

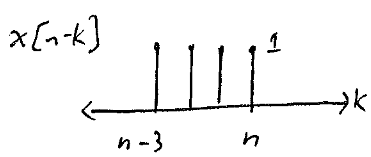
/Ex

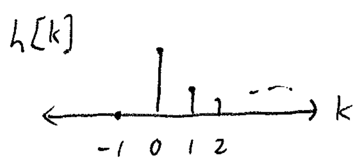
Ex $\quad x[n] = u[n] - u[n-4] \qquad h[n] = \left(\frac{5}{9}\right)^n u[n]$



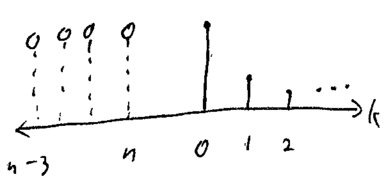


$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$



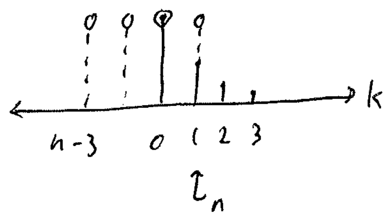


Case 1: $n < 0$



$y[n] = 0$

Case 2: $0 \le n \le 3$

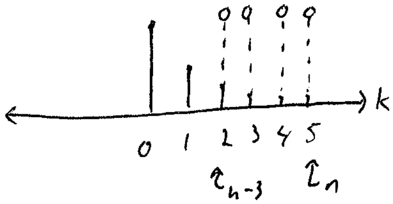


$$y[n] = \sum_{k=0}^{n} \left(\frac{5}{9}\right)^k (1)$$

$$= \frac{1-\left(\frac{5}{9}\right)^{n+1}}{1-\frac{5}{9}} = \frac{9}{4}\left(1-\left(\frac{5}{9}\right)^{n+1}\right)$$

Note: $\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r}$

Case 3: $n > 3$



$$y[n] = \sum_{k=n-3}^{n} \left(\frac{5}{9}\right)^k (1) \qquad \text{Let } m = k-(n-3)$$

$$= \sum_{m=0}^{3} \left(\frac{5}{9}\right)^{m+n-3}$$

$$= \left(\frac{5}{9}\right)^{n-3} \sum_{m=0}^{3} \left(\frac{5}{9}\right)^m = \left(\frac{5}{9}\right)^{n-3} \frac{1-\left(\frac{5}{9}\right)^4}{1-\frac{5}{9}}$$

$$= \frac{9}{4}\left(1-\left(\frac{5}{9}\right)^4\right)\left(\frac{5}{9}\right)^{n-3}$$

$$y[n] = \begin{cases} \frac{9}{4}\left(1-\left(\frac{5}{9}\right)^{n+1}\right) & 0 \le n \le 3 \\ \frac{9}{4}\left(1-\left(\frac{5}{9}\right)^4\right)\left(\frac{5}{9}\right)^{n-3} & n > 3 \\ 0 & \text{ow} \end{cases}$$

/Ex