What are signals & systems?

Signals: Any quantifiable source of information.
Typically indexed by time and/or space

Ex: Voltage on an antenna
temperatures across the country
blood pressure over time
engine torque and rpm
⋮

Continuous time: If the signal exists for all time ($t \in \mathbb{R}$) it is called a "continuous-time signal", $x(t)$

Discrete time: If the signal exists for discrete, integer-values of time ($n \in \mathbb{Z}$), it is called a "discrete-time signal", $x[n]$.

Continuous-Time	Discrete-Time
Temperature.	High temperatures measured each day.
Color Spectrum.	RGB values of a digital image.
Sound wave.	CDs.

System: Something that takes a signal as input and produces another signal as output.

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

Ex: full-wave rectifier:

$y(t) = |x(t)|$ $\qquad$ $y[n] = |x[n]|$

moving average:

$y(t) = \int_{t-1}^{t} x(\tau)\,d\tau$ $\qquad$ $y[n] = \frac{1}{10} \sum_{k=n-9}^{n} x[k]$

feedback:

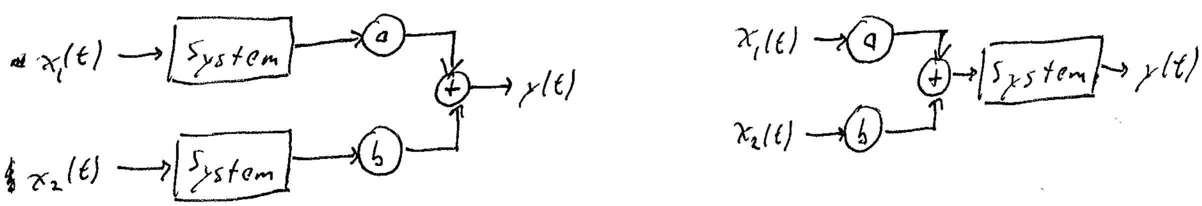
$y(t) = x(t) - \frac{1}{2} y(t-1)$ $\qquad$ $y[n] = x[n] - \frac{1}{2} y[n-1]$

/Ex

System Properties

Linearity

If a system is linear, the outputs below are always identical.



That is to say, for a linear system, $T(\cdot)$

Given $y_1(t) = T(x_1(t))$ & $y_2(t) = T(x_2(t))$

$T(a x_1(t) + b x_2(t)) = a y_1(t) + b y_2(t)$

Ex: $y(t) = x(t) + \frac{1}{2}x(t-2)$

So, $y_1(t) = x_1(t) + \frac{1}{2}x_1(t-2)$ & $y_2(t) = x_2(t) + \frac{1}{2}x_2(t-2)$

Let $x_3(t) = a\,x_1(t) + b\,x_2(t)$

What is $y_3(t)$?

$$\begin{aligned} y_3(t) &= x_3(t) + \tfrac{1}{2}x_3(t-2) \\ &= a\,x_1(t) + b\,x_2(t) + \tfrac{1}{2}\left(a\,x_1(t-2) + b\,x_2(t-2)\right) \\ &= a\left(x_1(t) + \tfrac{1}{2}x_1(t-2)\right) + b\left(x_2(t) + \tfrac{1}{2}x(t-2)\right) \\ &= a\,y_1(t) + b\,y_2(t) \end{aligned}$$

Therefore the system is linear.

/Ex

Ex

$$y(t) = \begin{cases} x(t) & x(t) \geq 0 \\ 0 & x(t) < 0 \end{cases}$$

This is not linear. We only need a single counter example.

Let $x_1(t) = 1$ & $x_2(t) = -1$

$y_1(t) = 1$ $\quad y_2(t) = 0$

Let $x_3(t) = x_1(t) + x_2(t) = 0$

$y_3(t) = 0 \neq y_1(t) + y_2(t)$

/Ex

Ex:
$$y[n] = \begin{cases} x[n] & 0 \le n \le 4 \\ 0 & ow \end{cases}$$

Let $y_1[n] = \begin{cases} x_1[n] & 0 \le n \le 4 \\ 0 & ow \end{cases}$ & $y_2[n] = \begin{cases} x_2[n] & 0 \le n \le 4 \\ 0 & ow \end{cases}$

$$x_3[n] = \alpha x_1[n] + \beta x_2[n]$$

$$y_3[n] = \begin{cases} x_3[n] & 0 \le n \le 4 \\ 0 & ow \end{cases}$$

$$= \begin{cases} \alpha x_1[n] + \beta x_2[n] & 0 \le n \le 4 \\ 0 & ow \end{cases}$$

$$= \alpha \begin{cases} x_1[n] & 0 \le n \le 4 \\ 0 & ow \end{cases} + \beta \begin{cases} x_2[n] & 0 \le n \le 4 \\ 0 & ow \end{cases}$$

$$= \alpha y_1[n] + \beta y_2[n]$$

/Ex Therefore, the system is linear.

Ex $y[n] = (x[n])^2$

this system is non-linear.

Let $x_1[n] = 1$ & $x_2[n] = -1$

$y_1[n] = 1$ $\quad$ $y_2[n] = 1$

Let $x_3[n] = x_1[n] + x_2[n] = 0$

$y_3[n] = (x_3[n])^2 = 0 \neq y_1[n] + y_2[n]$

/Ex

Time - Invariance

If a time-shift in the input causes an identical shift in the output, the system is said to be "time-invariant"

Given $x(t) \rightarrow \boxed{\text{System}} \rightarrow y(t)$,

Does $x(t-t_0)$ result in the output being $y(t-t_0)$?

Ex: $y(t) = |x(t)|$

putting $x(t-t_0)$ into the system yields output $|x(t-t_0)|$

$y(t-t_0) = |x(t-t_0)| \Leftrightarrow$ shifting the output yields $|x(t-t_0)|$

/Ex Thus, the system is time-invariant.

Ex $y(t) = x(2t)$

putting $x(t-t_0)$ into the system yields output $x(2(t-t_0))$

shifting the output yields $x(2t - t_0) \rightarrow \neq$

/Ex Thus, the system is time-varying

Ex $y[n] = \begin{cases} x[n] & x[n] \geq 0 \\ 0 & ow \end{cases}$

putting $x[n-n_0]$ into the system yields $\begin{cases} x[n-n_0] & x[n-n_0] \geq 0 \\ 0 & ow \end{cases}$

shifting the output yields $\begin{cases} x[n-n_0] & x[n-n_0] \geq 0 \\ 0 & ow \end{cases}$

/Ex Therefore the system is time-invariant.

Ex $y[n] = (-1)^n x[n]$

Let $x[n] = (-1)^n$, $y[n] = 1$

$x[n-1]$ yields $(-1)^n (-1)^{n-1} = -1$

$y[n-1] = 1 \longrightarrow \neq$

/Ex Therefore the system is time-varying.

<u>Causality</u>

If a system uses only past & present values of the input to compute the output, it is called a "causal" system.

If it uses future values it is "non-causal".

Ex: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

/Ex this system is causal because it never uses future values.

Ex $y(t) = x(2t)$

/Ex $y(1) = x(2) \Rightarrow$ future values are required $\Rightarrow$ non-causal

Ex: $y[n] = x[n] - \frac{1}{2} y[n-2]$

/Ex causal

Ex $y[n] = x[-n]$

/Ex $y[-1] = x[1] \Rightarrow$ non-causal

Memoryless

If a system depends only on the current input (not past or future) it is called "memoryless".

Ex $y(t) = x^3(t)$

/Ex memoryless

Ex $y(t) = \frac{1}{2}\int_{t-2}^{t} x(\tau)\,d\tau$

/Ex has memory

Ex $y[n] = \cos(2n)\,x[n]$

/Ex memoryless

Ex $y[n] = x[n] - \frac{1}{4}y[n-1]$

has memory

$$y[1] = x[1] - \frac{1}{4}y[0]$$
$$= x[1] - \frac{1}{4}\left(x[0] - \frac{1}{4}y[-1]\right)$$
$$\vdots$$

/Ex

Stability

A system that takes in a bounded input & produces a bounded output is called "stable".

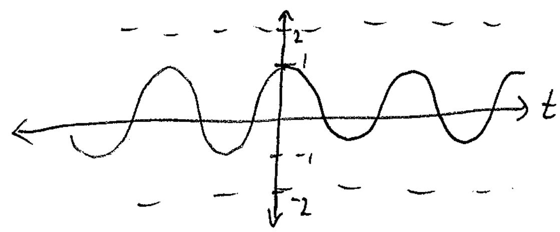
If it is possible for it to output an unbounded signal when the input is bounded, it is called "unstable".

Bounded:

If it is possible to choose a value $B \in \mathbb{R}$ such that $|x(t)| < B$ for all $t \in \mathbb{R}$, then $x(t)$ is bounded

Ex: $x(t) = \cos(t)$ is bounded

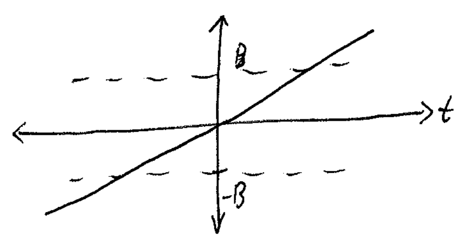
Let $B = 2$



/Ex

Ex $x(t) = t$

No matter what value of B you choose, there are values of t for which $|x(t)| > B \Rightarrow$ unbounded



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Stability Examples:

Ex $y(t) = 4\cos(\pi t)\, x(t)$

Assume $|x(t)| < B$

$$|y(t)| = 4|\cos(\pi t) x(t)|$$
$$\le 4|\cos(\pi t)||x(t)|$$
$$\le 4(1)|x(t)| \le 4B$$

$\Rightarrow$ $y(t)$ is bounded

/Ex Therefore, the system is stable.

Ex $y(t) = \frac{1}{x(t)}$

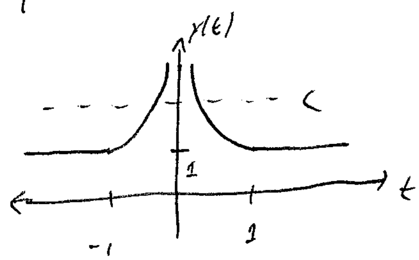
Let $x(t) = \begin{cases} |t| & |t| \le 1 \\ 1 & ow \end{cases}$

$|x(t)| \le 2$ for all t

Let $B = 2$

$y(t) = \begin{cases} \frac{1}{|t|} & |t| \le 1 \\ 1 & ow \end{cases}$

No matter what bound you choose $|y(t)| > C$



/Ex

Ex $y[n] = x^2[n]$

Assume $|x[n]| < B$

$|y[n]| = |x[n]|^2 < B^2$ $\Rightarrow$ the system is stable.

/Ex