Fourier Series

LTI systems & complex exponentials

We have shown that we can always express the output of an LTI system as a convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad\qquad y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Let's let the input be a complex exponential: $e^{j\omega t}$ or $e^{j\omega n}$

What is the output:

Continuous - Time:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau \\ &= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}_{H(j\omega)} = H(j\omega)\, e^{j\omega t} \end{aligned}$$

$H(j\omega)$ depends only on the impulse response and frequency (ω)

Discrete - Time:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)} \\ &= \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}}_{H(j\omega)}\, e^{j\omega n} \end{aligned}$$

Big picture Result: If you put in a signal with frequency ω to an LTI system, the output is also of frequency ω!

Harmonically Related Complex Exponentials

Continuous - Time

Let $\phi_k(t) = e^{jk\omega_0 t}$ be a family of complex exponentials that is fixed for a given ω_0

Ex: $\omega_0 = 100$

$\phi_0(t) = 1$

Ex $\phi_1(t) = e^{j100t}$ $\qquad \phi_{-1}(t) = e^{-j100t}$ etc.

Property 1: All of ϕ_k's have the same period

Let $T = \frac{2\pi}{\omega_0}$

$$\phi_k(t+T) = e^{jk\omega_0(t+T)}$$
$$= \left(e^{jk\omega_0 t}\right)\left(e^{jk\omega_0 T}\right)$$
$$= \left(e^{jk\omega_0 t}\right)\left(e^{jk\omega_0 \frac{2\pi}{\omega_0}}\right) = e^{jk\omega_0 t} = \phi_k(t)$$

Property 2: ϕ_k is orthogonal to ϕ_ℓ

$$\int_T \phi_k(t)\,\phi_\ell^*(t)\,dt = \int_{t_0}^{t_0+T} e^{jk\omega_0 t}\,e^{-j\ell\omega_0 t}\,dt$$
$$= \int_{t_0}^{t_0+T} e^{j\omega_0 t(k-\ell)}\,dt = \frac{1}{j\omega_0(k-\ell)}\,e^{j\omega_0 t(k-\ell)}\Big|_{t_0}^{t_0+T}$$
$$= \frac{1}{j\omega_0(k-\ell)}\left(e^{j\omega_0(k-\ell)(t_0+T)} - e^{j\omega_0(k-\ell)t_0}\right)$$
$$= \frac{1}{j\omega_0(k-\ell)}\,e^{j\omega_0(k-\ell)t_0}\left(e^{j(k-\ell)\omega_0\frac{2\pi}{\omega_0}} - 1\right) = 0$$

So, for any $k \neq \ell$ $\quad \int_T \phi_k(t)\,\phi_\ell^*(t)\,dt = 0$

Property 3: There are an infinite number of unique ϕ_k's since k can be any integer and each ϕ_k is orthogonal from all the others.

Periodic Signals can be expressed as linear combinations of Harmonically Related Complex Exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

since each $\phi_k(t)$ has period T, $x(t)$ is periodic with period T.

Finding a_k:

$$\int_T x(t) e^{-j\frac{2\pi}{T}\ell t}\,dt = \int_T \sum_{k=-\infty}^{\infty} a_k \phi_k(t)\phi_\ell^*(t)\,dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_T \phi_k(t)\phi_\ell^*(t)\,dt$$

Remember, if $k \neq \ell$ $\int_T \phi_k(t)\phi_\ell^*(t)\,dt = 0$

If $k=\ell$ $\int_T \phi_k(t)\phi_\ell^*(t)\,dt = \int_T e^{-jk\frac{2\pi}{T}t} e^{jk\frac{2\pi}{T}t}\,dt = T$

$$= \sum_{k=-\infty}^{\infty} a_k \, T\delta(k-\ell)$$

$$= T a_\ell \qquad \text{So} \qquad a_\ell = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi}{T}\ell t}\,dt$$

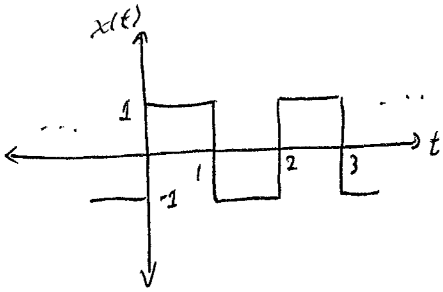
Fourier Series Equations:

☆ $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi}{T}kt}\,dt$$ ☆

Intuition

Each harmonic (value of $\pm k$) combines to fill out shape of the signal.

Ex:



period $T = 2$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{2}\int_0^2 x(t)\, e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\left[\int_0^1 (1)\, e^{-jk\pi t}\, dt + \int_1^2 (-1)\, e^{-jk\pi t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{-1}{jk\pi} e^{-jk\pi t}\Big|_0^1 + \frac{1}{jk\pi} e^{-jk\pi t}\Big|_1^2\right]$$

$$= \frac{1}{2}\,\frac{1}{jk\pi}\left[-\left(e^{-jk\pi} - 1\right) + \left(e^{-jk2\pi} - e^{-jk\pi}\right)\right]$$

$$= \frac{1}{j2k\pi}\left(-(-1)^k + 1 + 1 - (-1)^k\right) = \frac{1 - (-1)^k}{jk\pi} = \begin{cases} 0 & k \text{ even} \\ \frac{2}{jk\pi} & k \text{ odd} \end{cases}$$

Remember, $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\frac{2\pi}{T}t}$

$$= \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k}{jk\pi}\, e^{jk\pi t}$$

Let's see what happens as we use more and more harmonics to construct $x(t)$.

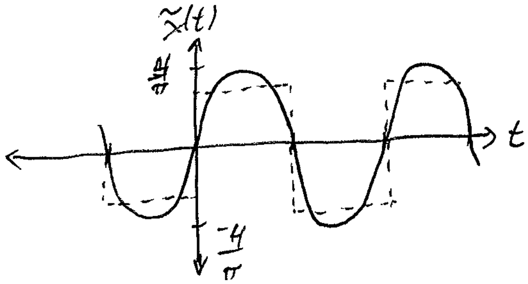
(more harmonics $\Longleftrightarrow$ higher values of k)

$$\tilde{x}(t) = \sum_{k=-N}^{N} \frac{1 - (-1)^k}{jk\pi}\, e^{jk\pi t}$$

$N = 1$:

$$\tilde{x}(t) = \frac{2}{j1\pi} e^{j1\pi t} + \frac{2}{j(-1)\pi} e^{j(-1)\pi t}$$

$$= \frac{2}{j\pi}\left(e^{j\pi t} - e^{-j\pi t}\right) = \frac{2}{j\pi} 2j \sin(\pi t) = \frac{4}{\pi} \sin(\pi t)$$



Already, we can see that
$\tilde{x}(t) \approx x(t)$
As we add more harmonics, $\tilde{x}(t)$ will get closer and closer to $x(t)$

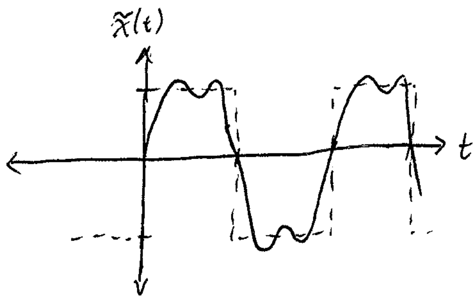
$N = 3$:

$$\tilde{x}(t) = \frac{4}{\pi} \sin(\pi t) + \frac{2}{j3\pi} e^{j3\pi t} + \frac{2}{j(-3)\pi} e^{j(-3)\pi t}$$

$$= \frac{4}{\pi} \sin(\pi t) + \frac{2}{j3\pi}\left(e^{j3\pi t} - e^{-j3\pi t}\right)$$

$$= \frac{4}{\pi} \sin(\pi t) + \frac{2}{j3\pi} 2j \sin(3\pi t)$$

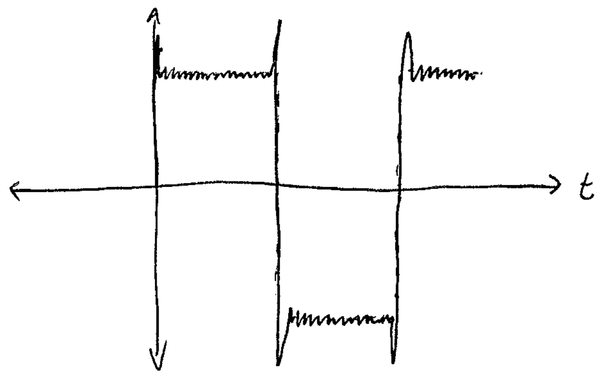
$$= \frac{4}{\pi} \sin(\pi t) + \frac{4}{3\pi} \sin(3\pi t)$$



/Ex

as the number of harmonics goes to ∞, $\tilde{x}(t)$ converges to $x(t)$ except for locations of discontinuities.

Gibbs Phenomenon:



At discontinuities you get overshoot beyond $x(t)$.

We're engineers, so we neglect this issue.

Discrete - Time

Harmonically Related Complex Exponentials

Let $\phi_k[n] = e^{jk\omega_0 n}$ be a family of complex exponentials that is fixed for a given ω_0

Ex: $\omega_0 = \frac{2\pi}{5}$

$\phi_0[n] = 1$

Ex $\phi_1[n] = e^{j\frac{2\pi}{5}n}$ $\qquad \phi_{-1}[n] = e^{-j\frac{2\pi}{5}n}$

Property 1: All of $\phi_k[n]$'s have same period

Let $N = 7 \qquad \omega_0 = \frac{2\pi}{7}$

$$\phi_k[n+N] = e^{jk\omega_0[n+N]} = e^{jk\frac{2\pi}{7}n} e^{jk\frac{2\pi}{7}7}$$

$$= e^{jk\frac{2\pi}{7}n}(1) = \phi_k[n]$$

Property 2: $\phi_k[n]$ is orthogonal of $\phi_\ell[n]$ for $k \neq \ell$

$$\sum_{n=0}^{N-1} \phi_k[n]\,\phi_\ell^*[n] = \sum_{n=0}^{N-1} e^{jk\omega_0 n} e^{-j\ell\omega_0 n}$$

$$= \sum_{n=0}^{N-1} e^{j(k-\ell)\omega_0 n} = \frac{1 - e^{j(k-\ell)\omega_0 N}}{1 - e^{j(k-\ell)\omega_0}}$$

$$= \frac{1 - e^{j(k-\ell)\frac{2\pi}{N}N}}{1 - e^{j(k-\ell)\omega_0}} = 0$$

Property 3: There are only N (finite!) unique $\phi_k[n]$'s.

Let's compare $\phi_{k+N}[n]$ & $\phi_k[n]$

$$\phi_{k+N}[n] = e^{j(k+N)\omega_0 n}$$
$$= e^{jk\omega_0 n} e^{jN\frac{2\pi}{N}n} = e^{jk\omega_0 n} = \phi_k[n]$$

So, $\phi_{k+N}[n]$ and $\phi_k[n]$ are NOT different.

Ex: $N = 5$

$$\phi_1[n] = e^{j\frac{2\pi}{5}n}$$
$$\phi_6[n] = e^{j\frac{2\pi}{5}(6)n} = e^{j\frac{12\pi}{5}n}$$
$$= e^{j\left(\frac{12\pi}{5} - \frac{10\pi}{5}\right)n} = e^{j\frac{2\pi}{5}n}$$

/Ex

Periodic signals can be represented as a linear combination of harmonically related complex exponentials.

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} a_k \phi_k[n]$$

since each $\phi_k[n]$ is periodic with period N,
$x[n]$ is also periodic with period N.

Finding a_k:

$$\sum_{n=0}^{N-1} x[n]\, e^{-jl\frac{2\pi}{N}n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k\, e^{jk\frac{2\pi}{N}n}\, e^{-jl\frac{2\pi}{N}n}$$

$$= \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n}\, e^{-jl\frac{2\pi}{N}n}$$

$$= \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} \phi_k[n]\, \phi_l^*[n]$$

Remember, if $k \neq l$ $\sum_{n=0}^{N-1} \phi_k[n]\, \phi_l^*[n] = 0$

If $k = l$ $\sum_{n=0}^{N-1} \phi_k[n]\, \phi_l^*[n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}kn} = N$

$$= \sum_{k=0}^{N-1} a_k\, N\, \delta[k-l]$$

$$= N a_l$$

so, $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$

Fourier Series Equations

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{jk\frac{2\pi}{N}n} \qquad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Intuition:

The discrete-time Fourier Series works exactly as the continuous-time: adding more harmonics gets you closer to the actual original signal, $x[n]$

Also, there is no Gibbs Phenomenon because there are no discontinuities in discrete-time.

This also means that you only need a finite number of terms to make $x[n]$ exactly.