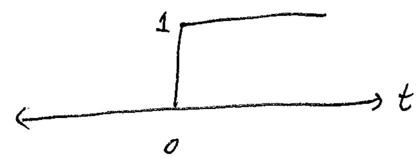
Unit Step and Impulse Functions

Continuous - Time

Unit-step function, $u(t)$

$$u(t) = \begin{cases} 1 & t<0 \\ 0 & t>0 \end{cases}$$

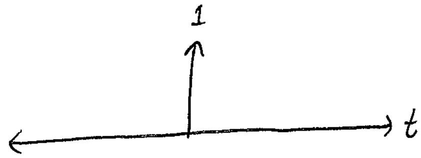


Note the discontinuity @ $t=0$.

Unit impulse function, $\delta(t)$

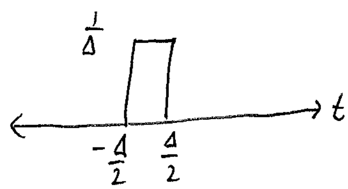
$$\delta(t) = \frac{d}{dt} u(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$



Other way to think of $\delta(t)$:

Let $$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & |t| < \frac{\Delta}{2} \\ 0 & \text{o w} \end{cases}$$

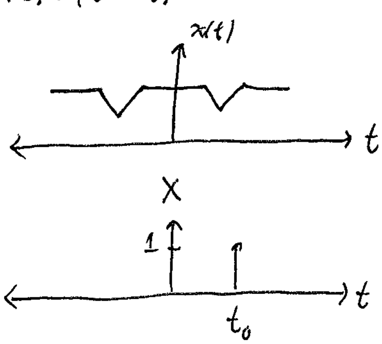


Note: the area of $\delta_\Delta(t)$ is always 1.

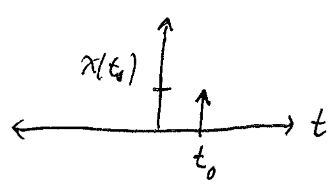
$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$: It gets infinitesimally ~~small~~ narrow and infinitely tall. But, the area is still 1!

Implications:

1) $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$



$\Longrightarrow$



we get a delta centered at t_0 with height $x(t_0)$

2) $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$ since the area of $\delta(t)$ is 1.

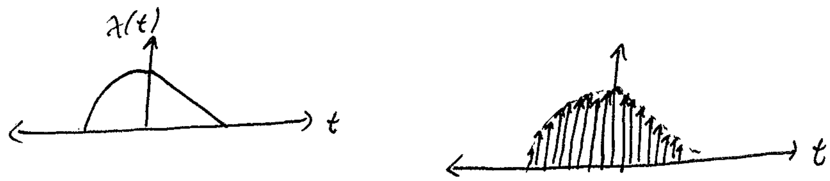
3) $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt$

$= x(t_0)$

To put it another way:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

This means we can think of $x(t)$ as the linear combination of a bunch of shifted and scaled $\delta(t)$'s.

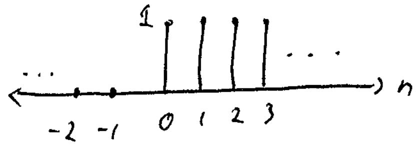


This insight will be important later (impulse response for LTI systems).

Discrete-Time

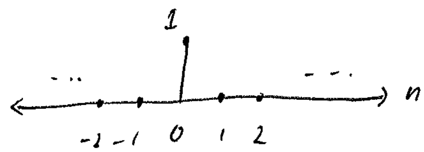
Unit Step Function, $u[n]$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Unit delta function, $\delta[n]$

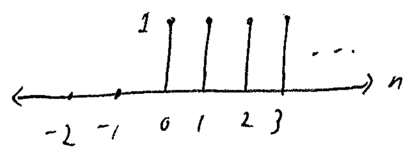
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Note that these definitions are much easier to understand.

Relationship of $\delta[n]$ & $u[n]$:

$$\delta[n] = u[n] - u[n-1]$$



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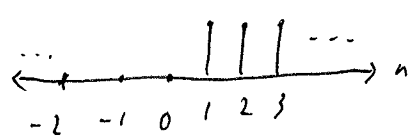

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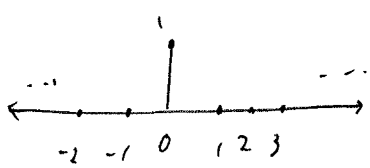
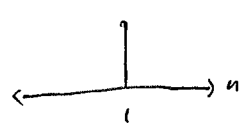


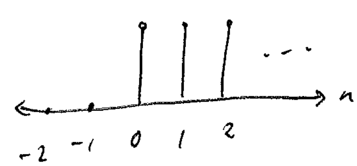
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

$$= \delta[n] + \delta[n-1] + \cdots$$

$+$

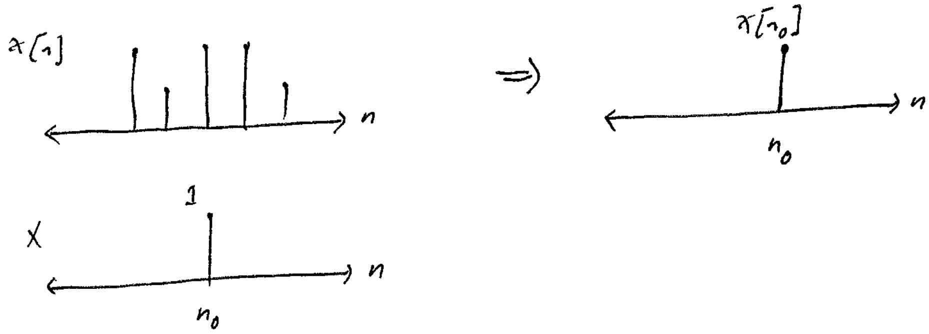


$+ \cdots =$



Implications of $\delta[n]$:

1) $x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$



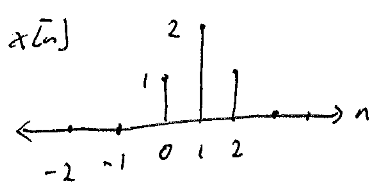
2) $\displaystyle\sum_{n=-\infty}^{\infty} \delta[n-n_0] = 1$

3) $\displaystyle\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_0] = \sum_{n=-\infty}^{\infty} x[n_0]\,\delta[n-n_0] = x[n_0]\sum_{n=-\infty}^{\infty} \delta[n-n_0] = x[n_0]$

Put another way:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$$

This means $x[n]$ really is a bunch of shifted and scaled $\delta[n]$'s.



$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$

this is an important concept and we will see it again when we get to discrete-time impulse response of LTI systems.