(15 pts) **1.** Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1].$$

Frequency shifting

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad e^{j\omega_0 n}x[n] = X e^{j(\omega-\omega_0)}$$

So let $x[n] = \left(\frac{1}{3}\right)^n u[n-1]$ and use frequency shift

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n-1] e^{-j\omega n}$$

$u[n-1] = 0 \qquad n-1 < 0$

$n < 1$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^n \qquad \text{let } m = n-1$$

$n = m+1$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^{m+1} \Rightarrow \frac{1}{3}e^{-j\omega} \sum_{m=0} \left(\frac{1}{3}e^{-j\omega}\right)^m$$

$$X(\omega) = \frac{1}{3}e^{-j\omega} \cdot \frac{1}{1-\frac{1}{3}e^{-j\omega}e^{-j\omega}}$$

then frequency shift by $\frac{\pi}{17}$

$$\boxed{X(\omega) = \frac{1}{3}e^{-j(\omega-\pi/17)} \cdot \frac{1}{1-\frac{1}{3}e^{-j}}}$$

(15 pts) **2.** Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$\mathcal{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi).$$

$$\mathcal{F}^{-1} = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{j\omega n} \underbrace{\sum_{k=0}^{\infty}}_{\text{depends on } k} \frac{1}{2^k} \delta(\omega - k\pi) d\omega$$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{2^k} \int_0^{2\pi} e^{j\omega n} \delta(\omega - k\pi) d\omega)$$

$$x[n] = \boxed{\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{2^k} e^{jk\pi n}}$$ by "sifting property"

x[n] is purely imaginary and ev

(cos ... ~~...~~ ~~[...]~~)

(15 pts) **3.** Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

X(ω) is odd and ...

Test if $X(\omega)$ is real or imag.

If real $X(\omega) = X^*(-\omega) = \frac{-j}{\sin -\omega} = \frac{-j}{-\sin\omega} = \frac{j}{\sin\omega}$

which indicates x[n] is real, but j cos(g[n]) is imaginary

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{3}{\cos\omega}$. Could Alice be right? Explain.

$X^*(-\omega) = \frac{3}{\cos -\omega} = \frac{3}{\cos\omega}$ so x[n] would be real, but j cos(g[is NOT, so Alice is not right

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{(\omega^2+1)^2}$. Could Devin be right? Explain.

$X^*(-\omega) = \frac{-j}{((-\omega)^2+1)^2} = \frac{-j}{(\omega^2+1)^2} \neq X(\omega)$ so x[n is not r

So might be correct

4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

$$x[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0} X(\omega)$$

$$Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

$$Y(\omega)\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right) = 2X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \boxed{\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}}$$

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$$h[n] = \mathcal{F}^{-1}(H(\omega))$$

$$\frac{2}{1 - 3/4 e^{-j\omega} + 1/8 e^{-j2\omega}} = \frac{A}{1 - 1/2 e^{-j\omega}} + \frac{B}{1 + 1/4 e^{-j\omega}}$$

$$= \frac{4}{1 - 1/2 e^{-j\omega}} - \frac{2}{1 - 1/4 e^{-j\omega}}$$

$$a^n u[n] \xrightarrow{\mathcal{F}} \frac{1}{1 - a e^{-j\omega}}$$

$$\boxed{h[n] = 4 \cdot \left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]}$$

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer.)

$$y[n] = x[n] * h[n]$$

or

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$X(\omega) = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

$$\boxed{Y(\omega) = X(\omega) \cdot H(\omega) = \frac{1}{1 - \frac{1}{4} e^{-j\omega}} \cdot \frac{2}{(1 - 3/4 e^{-j\omega} + \frac{1}{8} e^{-2j\omega})}}$$

$$= \frac{1}{\left(1 - \frac{1}{4} e^{-j\omega}\right)} \cdot \frac{2}{\left(1 - \frac{1}{4} e^{-j\omega}\right)\left(1 - 1/2 e^{-j\omega}\right)}$$

$$\boxed{Y(\omega) = \frac{2}{\left(1 - \frac{1}{4} e^{-j\omega}\right)^2 \left(1 - \frac{1}{2} e^{-j\omega}\right)}}$$

(10 pts) d) What is the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer)

$$\mathcal{F}^{-1}(Y(\omega)) \Rightarrow \frac{A}{(1-\frac{1}{4}e^{-j\omega})} + \frac{B}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{C}{(1-\cdot}$$

$x = e^{-j\omega}$

$$\frac{2}{(1-\frac{1}{4}x)^2(1-\frac{1}{2}x)} = \frac{A}{1-\frac{1}{4}x} + \frac{B}{(1-\frac{1}{4}x)^2} + \frac{C}{(1-\frac{1}{2}x}$$

$B = -2 \qquad C = 8 \qquad A = -4$

$$\boxed{y[n] = -4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n] + 8(}$$

(15 pts) **5.** Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2} dt.$$

Use property 20:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\sin(4t)}{t}\right|^2 dt \qquad \text{need } \mathcal{F}\left(\frac{\sin(4t)}{t}\right)$$

using #5: $\frac{\sin Wt}{\pi t} \xrightarrow{\mathcal{F}} u(\omega+W) - u(\omega-W)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\pi \sin(4t)}{\pi \, t}\right|^2$$

?? stuck