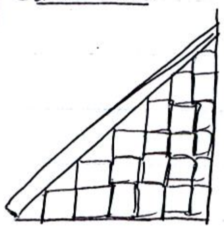
21 February 2012

Last Time



a_n = # walks along the diagonal

challeng: The sequence turns out to be a very famous one.

$\exists$ book showing that this sequence arises in several hundred ways.

e.g.

If you multiply $n+1$ numbers together

$\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_n$

the sequence of multiplication may vary:

$$(((\lambda_0 \lambda_1) \lambda_2(\lambda_3 \lambda_4)))\ldots \lambda_n$$

placement of parentheses clarify the order of multiplication.

$n=1$: $(\lambda_0 \lambda_1)$ $\iff 1$

$n=2$: $(\lambda_0 \lambda_1)\lambda_2$: $\lambda_0(\lambda_1 \lambda_2)$ $\iff 2$

$n=3$: $((\lambda_0\lambda_1)(\lambda_2 \lambda_3))$ $(((\lambda_0 \lambda_1)\lambda_2)\lambda_3)$ $((\lambda_0(\lambda_1 \lambda_2))\lambda_3)$ $(\lambda_0((\lambda_1 \lambda_2)\lambda_3))$ $\iff 5$.

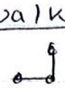
$(\lambda_0(\lambda_1(\lambda_2 \lambda_3)))$ ~~$(\lambda_0 \lambda_1(\lambda_2 \lambda_3))$~~

Let b_n = # of ways to multiply $n+1$ numbers

we show than $a_n = b_n$ ..

$\Rightarrow$ we provide translation as follows:

walk | patter

$n=1$  $(**)$ $\leftarrow$ (identity of λ does not really mater)

$n=2$  $((**)*)$, $(*(**))$

$n=3$  ...

(to be cont.)

Constructing Recursions

Q: Let a_n = # of 0/1 bitstring of

a. length n

b. length n w/ at least 1 zero

c. length n w/ at least 2 consective zero

Find recursions for each situations

We approach each situation by expressing bitstring as sequence of shorter ones:

A: (a) If we cut off the last bit:

0 ↙ ↘ 1

length $n-1$ → no other constraints $= a_{n-1}$

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thus $a_n = a_{n-1} + a_{n-1}$ initial condition: $a_1 = 2$ (can be 1 or 0)

(b) Cutting off the last digit

0 ↙ ↘ 1

legth $n-1$ no other constraints $\Updownarrow$ $a_{n-1} + 1$ (w/ zero ↙, w/o zero ↘)

length $n-1$ still need at least 1 zero $\Updownarrow$ a_{n-1}

$a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$

$a_1 = 1$ (0)

→ observe that a_n which describes the mechanism, indicates that (b) grows more rapidly than (a). Intuitively, we can conclude that (a) should always be greater than (b). This discrepancy is explained by the difference in initial condition

(c) cut off the last bit

0 ↙ ↘ 1

length $n-1$: 0 ↙ can be anything... 2^{n-2}; 1 ↘ still need 2 consec a_{n-2}

length $n-1$ still need conseq 0s $\Leftrightarrow a_{n-1}$

$a_n = a_{n-1} + a_{n-2} + 2^{n-2}$

two initial cond

$a_0 = 0$

$a_1 = 0$

Linear Recurrence of order K

$$a_n = c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \dots + c_{n-k} a_{n-k} + F(n)$$

where $c_{n-1} \dots c_{n-k} \in \mathbb{R}$

$F(n)$ will be a sum of terms from following dist.

→ a polynomial in n

→ λ^n, ..., $\lambda \in \mathbb{R}$

→ combination of above two.

If $F(n) = 0$, the reurrence is called homogeneous.

The homogeneous LRoOK

(*) $a_n = c_{n-1} a_{n-1} + \dots + c_{n-k} a_{n-k}$

Def characteristic equation of (*) is

$$\lambda^n = c_{n-1} \lambda^{n-1} + c_{n-2} \lambda^{n-2} + \dots + c_{n-k} \lambda^{n-k} \quad : \text{Divide by } \lambda^{n-k}$$

$$\lambda^k = c_{n-1} \lambda^{k-1} + c_{n-2} \lambda^{k-2} + \dots c_{n-k}$$

and the characteristic roots are the values $\lambda_1, \lambda_2, \dots \lambda_k$ that makes above equation true.

Thm For homogeneous LRoOK all solutions are linear combination of terms $\lambda_1^n, \lambda_2^n, \dots, \lambda_k^n$.

(in fact, they form the basis for the solution space)

Ex: Recall Rabbits!

$$a_n = a_{n-1} + a_{n-2} \qquad a_1 = 1 \qquad a_2 = 2$$

characteristic equation:

$$\lambda^k = c_{n-1} \lambda^{k-1} + c_{n-2} \lambda^{k-2} \dots$$

$$\lambda^2 = (1)(\lambda^{2-1}) + (1) \lambda^{2-2}$$

$$\lambda^2 = \lambda + 1$$

$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow$ apply our theorem, all of the solutions to $a_n = a_{n-1} + a_{n-2}$ looks like

$$b_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + b_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

our initial conditions say:

$n=1$

$$1 = b_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + b_2 \left(\frac{1-\sqrt{5}}{2}\right)^1$$

$n=2$

$$2 = b_1 \left(\frac{1+\sqrt{5}}{2}\right)^2 + b_2 \left(\frac{1-\sqrt{5}}{2}\right)^2$$

and some nasty calculation will follow.

ex 2 : $a_n = 2a_{n-1} - a_{n-2}$

characteristic eq.

$$\lambda^2 = 2\lambda - 1$$

$$\lambda_{1,2} = 1, 1$$

According to our theorem, all solutions are linear combinations of 1^n and 1^n i.e. constant.

This make sense; if $a_n = a_{n-1} = a_{n-2}$, $a_n = 2a_{n-1} - a_{n-2} =$
$= 2a_n - a_n = a_n$ ✓

Now consider initial conditions

$a_0 = 0$

$a_1 = 1$ ..

no constants can be 0 and 1. what has happened?

? can initial condition be anything?