Problem 1

(a) $x[n] = 1, 1, 1, 1$

$$X(e^{j\omega}) = \sum_{n=0}^{3} 1 \cdot e^{-j\omega n} = \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}} \qquad \omega \neq 0$$

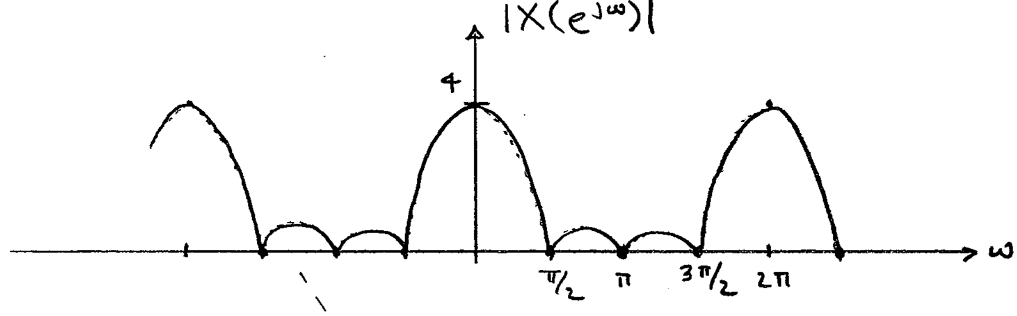
For later plotting it is helpful to write this in the "discrete time sinc form":

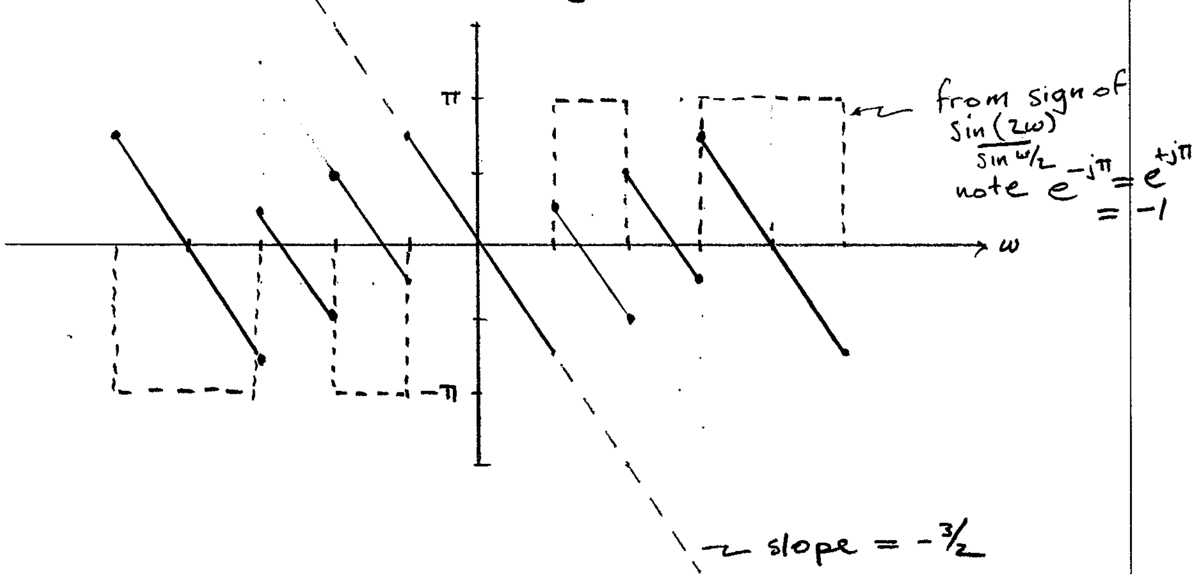
$$X(e^{j\omega}) = \frac{e^{-j2\omega}\left(e^{+j2\omega} - e^{-j2\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= e^{-j3\omega/2} \, \frac{\sin(2\omega)}{\sin(\omega/2)} \qquad \omega \neq 0, \pm 2\pi, \ldots$$

$$= 4 \qquad \omega = 0, \pm 2\pi \ldots$$

(b) zeros of $\sin 2\omega$ @ $2\omega = \pi k \Rightarrow \omega = \frac{\pi}{2} k$





(c) $$\tilde{X}_k = \frac{1}{N} X(e^{j\omega})\Big|_{\omega = 2\pi k/N}$$

Let $\tilde{x}[n]$ be the inverse DTFS of $\tilde{X}_k$ ie

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}_k e^{j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j2\pi k/N}) e^{j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi km/N}\right) e^{j2\pi kn/N}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N}}_{= \begin{cases} 1 & \text{if } m = n+\ell N \\ 0 & \text{otherwise} \end{cases}}$$

∴ the only nonzero terms in the infinite sum over m are

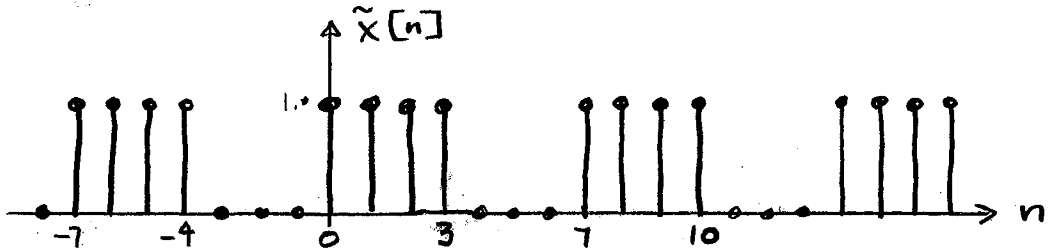
$$m = \dots\ n-2N,\ n-N,\ n,\ n+N,\ n+2N\ \dots$$

By re-indexing the sum we conclude

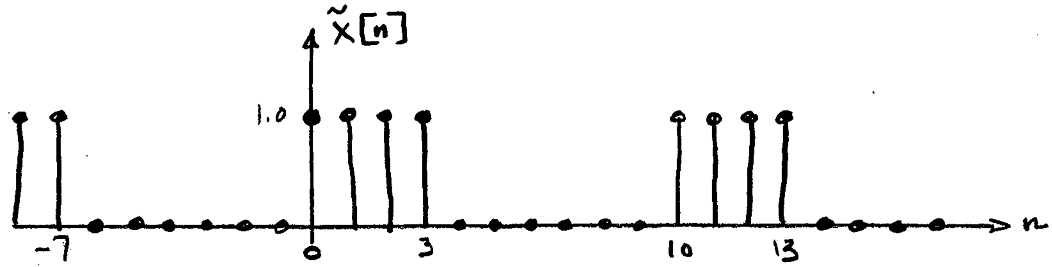
$$\tilde{x}[n] = \sum_{\ell=-\infty}^{\infty} x[n+\ell N] \qquad (*)$$

This holds for a general signal $x[n]$ and, in particular, for the finitely supported one considered here. If $x[n]$ is finitely supported, then for a fixed n, the sum in (*) would have only a finite number of nonzero terms.

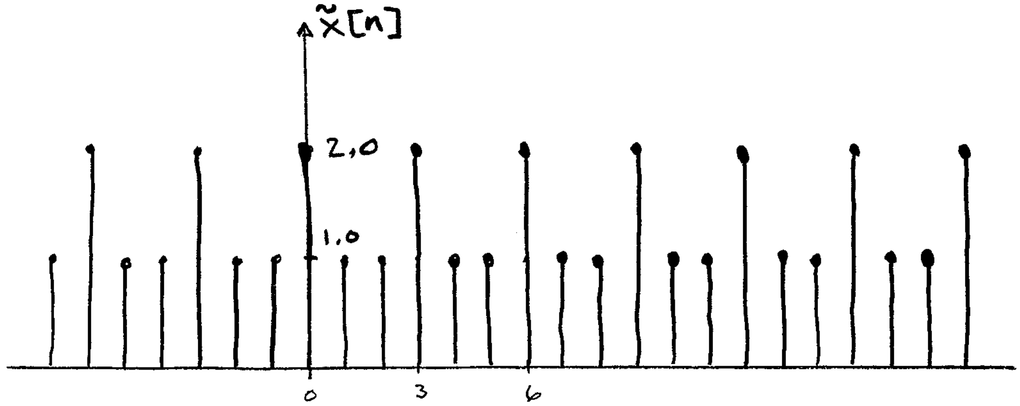
(c-i) $N=7$ $\Rightarrow$ $\tilde{x}[n]$ is simply $x[n]$ periodically extended with period $=7$.



(c-ii) $N=10$



(c-iii) $N=3$ $\Rightarrow$ will now have overlaps of the copies of $x[n]$ ie aliasing in time domain



Problem 2

(a)

$$\overline{X}_k = \frac{1}{2N}\sum_{n=0}^{2N-1} x[n]\, e^{-j2\pi kn/2N}$$

$$= \frac{1}{2N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/2N} + \frac{1}{2N}\sum_{n=N}^{2N-1} x[n]\, e^{-j2\pi kn/2N}$$

make c.o.v. in this sum — let $n' = n - N$ — then second sum

$$= \frac{1}{2N}\sum_{n'=0}^{N-1} x[n'+N]\, e^{-j2\pi kn'/2N}\, e^{-j2\pi kN/2N}$$

Then since $x[\cdot]$ is periodic with period N and $e^{-j2\pi kN/2N} = e^{-j\pi k} = (-1)^k$ can write the second sum as

$$= \left(\frac{1}{2N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/2N}\right)(-1)^k$$

Putting the two parts back together

$$\overline{X}_k = \left(\frac{1}{2N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/2N}\right)\underbrace{\left(1 + (-1)^k\right)}_{= \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}}$$

Thus let $k = 2\ell$

$$\overline{X}_{2\ell} = \left(\frac{1}{2N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi 2\ell n/2N}\right)(2)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \ell n/N} = X_\ell$$

Summarizing

$$\overline{X}_k = \begin{cases} X_{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

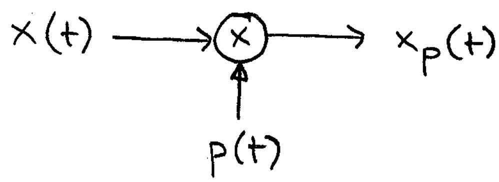
(b) The same type of argument (but a bit more complicated) can be used to show

$$\tilde{X}_k = \begin{cases} X_{k/K} & \text{if } k \text{ a multiple of } K \\ 0 & \text{otherwise} \end{cases}$$

Problem 3

(a) Rederive $X_p(j\omega) = \frac{1}{T}\sum_k X(j(\omega - k\omega_s))$.

From



and the multiplication in time domain property of the Fourier transform we know

$$x_p(t) = x(t)p(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega) = X_p(j\omega)$$

From the table of Fourier transforms

$$P(j\omega) = \frac{2\pi}{T}\sum_k \delta\left(\omega - \frac{2\pi}{T}k\right) \qquad \omega_s = \frac{2\pi}{T}$$

$$\therefore\ X_p(j\omega) = 2\pi\ X(j\omega) * \frac{2\pi}{T}\sum_k \delta(\omega - k\omega_s)$$

$$= \frac{1}{T}\sum_k X(j\omega) * \delta(\omega - k\omega_s)$$

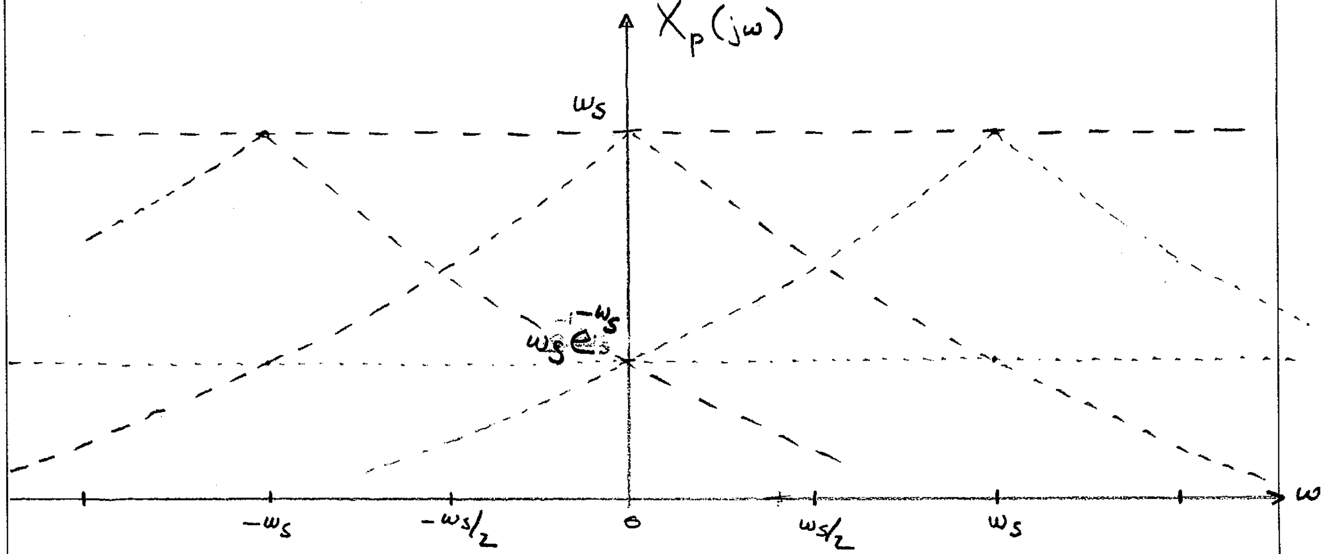
$$= \frac{1}{T}\sum_k X(j(\omega - k\omega_s))$$

(b) Now $X(j\omega) = 2\pi e^{-|\omega|}$; $\frac{1}{T} = \frac{\omega_s}{2\pi}$

$$X_p(j\omega) = \frac{\omega_s}{2\pi}\sum_k 2\pi e^{-|\omega - k\omega_s|}$$

$$= \omega_s \sum_k e^{-|\omega - k\omega_s|}$$

The plot we seek is the sum of the overlapping spectral components shown below with dashed lines



There are an infinite number of overlapping terms. To make a fairly careful plot only a few terms are really needed. The cases are

(b-i) $e^{-\omega_s} = 0.01$

(b-ii) $e^{-\omega_s} = 0.25$ (ω_s is larger in the first case).

For the plots it is helpful to approx. the values of $X_p(j\omega)$ at a few points:

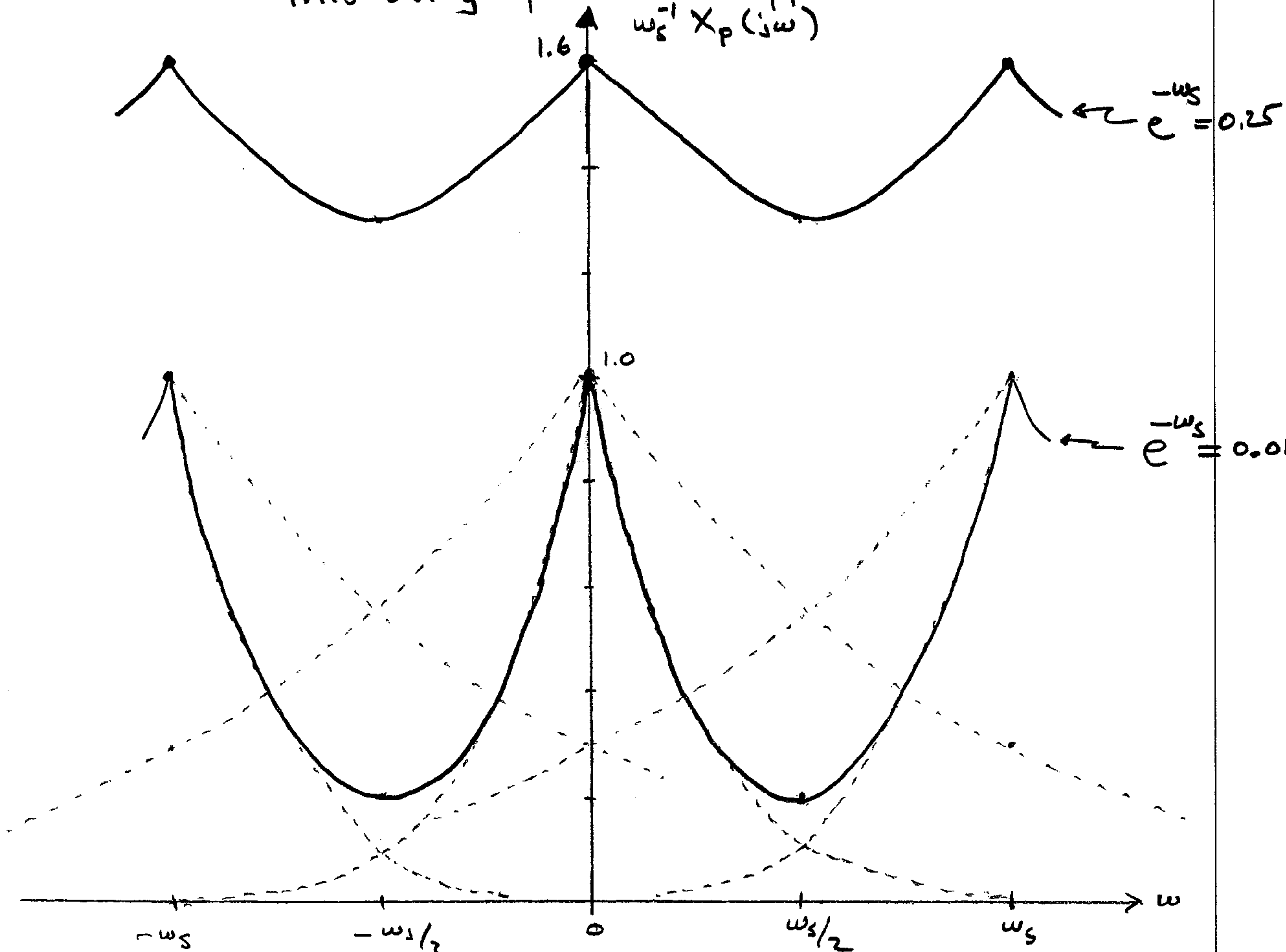
$$X_p(j\omega)\Big|_{\omega=0} \cong \omega_s + 2\omega_s e^{-\omega_s} + 2\omega_s e^{-2\omega_s}$$

$$\cong \omega_s\left(1 + 2e^{-\omega_s} + 2e^{-2\omega_s}\right)$$

$$X_p(j\omega)\Big|_{\omega=\omega_s/2} \approx 2\omega_s e^{-\omega_s/2} + 2\omega_s e^{-3\omega_s/2}$$

	$\omega_s^{-1} X_p(j\omega)\big|_{\omega=0}$	$\omega_s^{-1} X_p(j\omega)\big|_{\omega=\omega_s/2}$
$e^{-\omega_s} = 0.01$	1.0202	0.202
$e^{-\omega_s} = 0.25$	1.625	1.25

this using previous approximations.



From picture clear that aliasing distortion is more severe in the case where

$$e^{-\omega_s} = 0.25.$$

(c) $$X_r(j\omega) = \begin{cases} \sum_k X(j(\omega - k\omega_s)) & -\omega_s/2 \le \omega \le \omega_s/2 \\ 0 & \text{else.} \end{cases}$$

So for $-\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2}$

$$X_r(j\omega) = 2\pi \sum_k e^{-|\omega - k\omega_s|}$$

$$= \underbrace{2\pi e^{-|\omega|}}_{\text{desired}} + \underbrace{2\pi \sum_{k=-\infty}^{-1} e^{-(\omega - k\omega_s)} + 2\pi \sum_{k=1}^{\infty} e^{-(k\omega_s - \omega)}}_{\text{aliased}}$$

$$= 2\pi e^{-|\omega|} + 2\pi e^{-\omega} \sum_{k=-\infty}^{-1} e^{k\omega_s} + 2\pi e^{\omega} \sum_{k=1}^{\infty} e^{-k\omega_s}$$

(d)

$$= 2\pi e^{-|\omega|} + 2\pi \left(e^{-\omega} + e^{\omega}\right) \sum_{k=1}^{\infty} e^{-k\omega_s}$$

$$= 2\pi e^{-|\omega|} + 2\pi \left(e^{-\omega} + e^{\omega}\right) \left[\frac{1}{1 - e^{-\omega_s}} - 1\right]$$

$$= \underbrace{2\pi e^{-|\omega|}}_{\text{desired}} + \underbrace{\frac{2\pi e^{-\omega_s}}{1 - e^{-\omega_s}} \left(e^{-\omega} + e^{\omega}\right)}_{\text{aliased}}$$

$$\therefore \quad X_{desired}(j\omega) = \begin{cases} 2\pi e^{-|\omega|} & |\omega| < \omega_s/2 \\ 0 & \text{else} \end{cases}$$

$$X_{aliased}(j\omega) = \begin{cases} \frac{2\pi e^{-\omega_s}}{1 - e^{-\omega_s}} \left(e^{-\omega} + e^{\omega}\right) & |\omega| < \omega_s/2 \\ 0 & \text{else.} \end{cases}$$

(e) $e^{-\omega_s} = 0.01 \implies \dfrac{e^{-\omega_s}}{1-e^{-\omega_s}} = \dfrac{0.01}{0.99}$

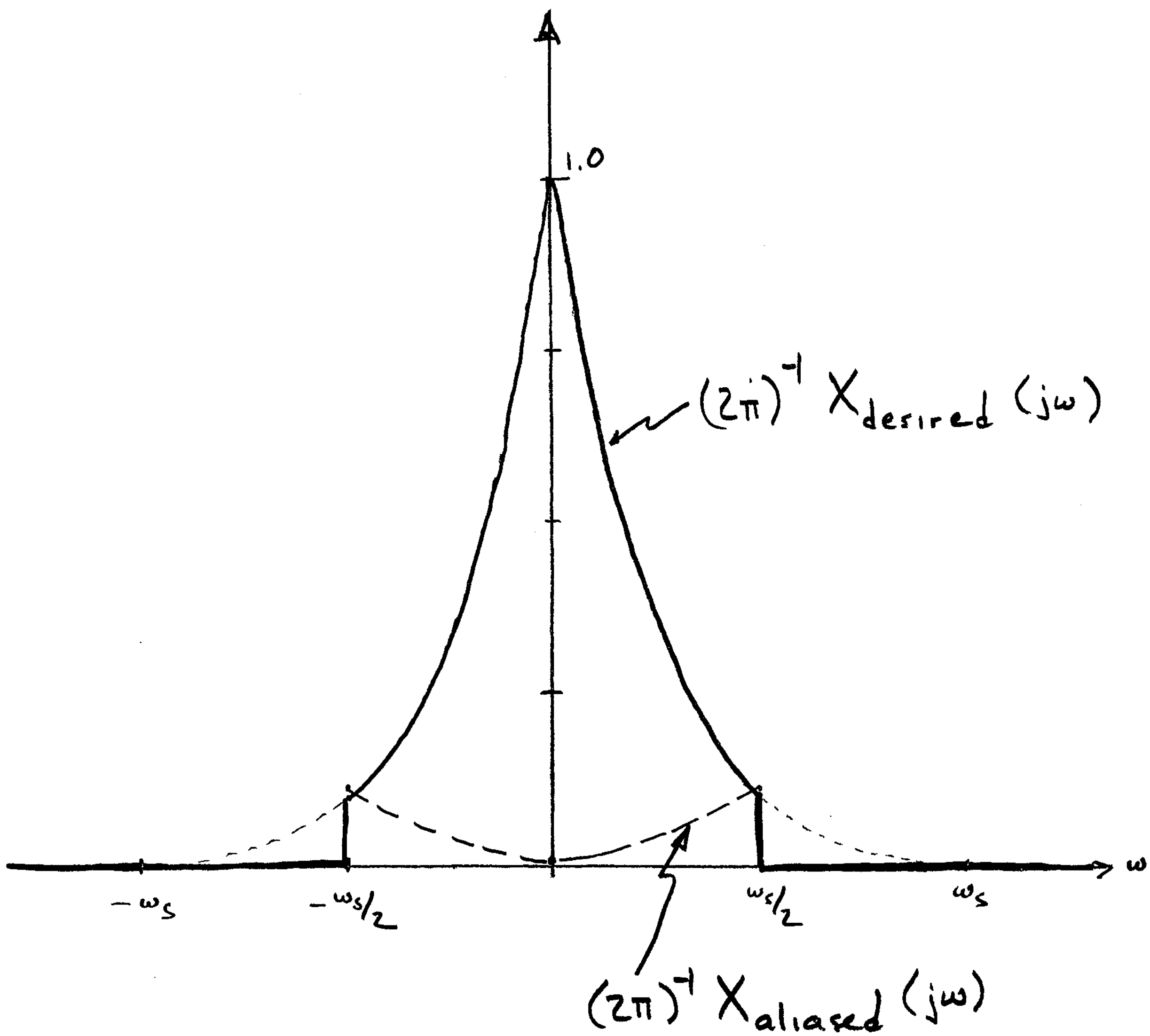
$$= \frac{1}{100} \cdot \frac{100}{99} = \frac{1}{99}$$

$$\therefore \quad (2\pi)^{-1} X_{desired}(j\omega) = 1 \quad @ \quad \omega = 0$$

$$= 0.1 \quad @ \quad \omega = \omega_s/2$$

$$(2\pi)^{-1} X_{aliased}(j\omega) = 2/99 \quad @ \quad \omega = 0$$

$$= \frac{101}{990} \quad @ \quad \omega = \omega_s/2$$



(f) $$E_{desired} = \frac{1}{2\pi}\int_{-\omega s/2}^{\omega s/2} \left(2\pi e^{-|\omega|}\right)^2 d\omega = 4\pi \int_0^{\omega s/2} e^{-2\omega}\, d\omega$$

$$= 4\pi \left(-\frac{1}{2}e^{-2\omega}\right)\Big|_{\omega=0}^{\omega s/2}$$

$$= 2\pi\left(1 - e^{-\omega_s}\right)$$

(Note that energy in original signal $x(t)$ is 2π).

$$E_{aliased} = \frac{1}{2\pi}(2\pi)^2\left(\frac{e^{-\omega_s}}{1-e^{-\omega_s}}\right)^2 \int_{-\omega s/2}^{\omega s/2} \underbrace{\left(e^{-\omega}+e^{\omega}\right)^2}_{= e^{-2\omega} + 2 + e^{2\omega}} d\omega$$

Compute the integral

$$\int_{-\omega s/2}^{\omega s/2} \left(e^{-2\omega} + 2 + e^{2\omega}\right) d\omega = -\frac{1}{2} e^{-2\omega}\Big|_{-\omega_s/2}^{\omega_s/2} + 2\omega_s + \frac{1}{2}e^{2\omega}\Big|_{-\omega s/2}^{\omega s/2}$$

$$= -\frac{1}{2}e^{-\omega s} + \frac{1}{2}e^{\omega s} + 2\omega_s + \frac{1}{2}e^{\omega_s} - \frac{1}{2}e^{-\omega s}$$

$$= 2\omega_s + e^{\omega_s} - e^{-\omega s}$$

$$\therefore\ E_{aliased} = 2\pi \frac{e^{-2\omega s}}{(1-e^{-\omega s})^2}\left[2\omega_s + e^{\omega s} - e^{-\omega s}\right].$$

$$= 2\pi\, \frac{2\omega_s e^{-2\omega_s} + e^{-\omega s} - e^{-3\omega s}}{(1-e^{-\omega s})^2}$$

Clearly

$$\lim_{\omega_s \to \infty} E_{desired} = 2\pi$$

$$\lim_{\omega_s \to \infty} E_{aliased} = 0$$

Note for $e^{-\omega_s} = 0.01$

$$E_{desired} = 2\pi(0.99)$$ ie 99% of orig. sig. energy.

$$E_{aliased} \approx 2\pi \frac{0.01}{(0.99)^2}$$ ie slightly more than 1% of orig. sig. energy.