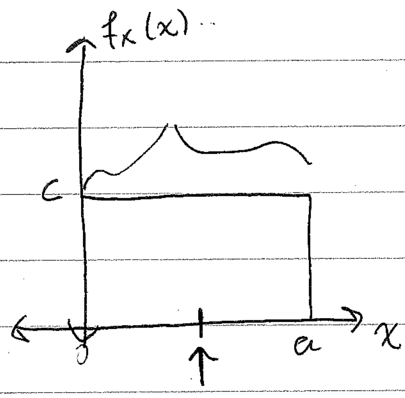
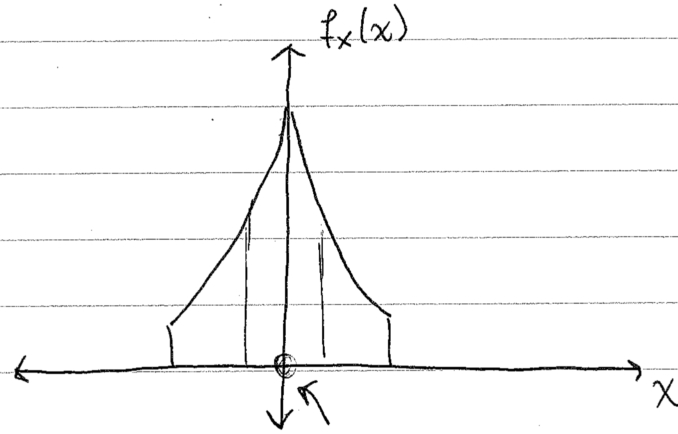
Expectation





Let X be ~~an~~ a r.v. Sometimes we desire only a few parameters to describe the behavior of X instead a cdf, pdf, or pmf. We will introduce parameters that quantify certain properties of r.v.s.

The expected value (mean) of r.v. X is

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \sum_{x_i} x_i\, P_X(x_i) \quad \text{if } X \text{ is discrete}$$

The expected value gives the average value seen when taking many observations of a r.v.

The n^{th} moment of a r.v. X is

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

$$= \sum_{x_i} x_i^n P_X(x_i) \quad \text{if } X \text{ is discrete}$$

The 2^{nd} central moment is called the variance of r.v. X and is given by

$$\mathrm{Var}[X] = \sigma_X^2 = E\left[(X - E[X])^2\right]$$

(spread)

The variance of X describes the width of the distribution of X

The <u>standard deviation</u> of X is

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

The expected value of a r.v. $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \sum_{x_i} g(x_i) \cancel{f_X} p_X(x_i)$$

<u>Note</u>: The mean, <u>n</u>th moment, <u>n</u>th central moment are all special cases of $\cancel{E[X^n]} = E[g(X)]$

Probability can also be interpreted as an expected value. Let $A \subset \mathbb{R}$ and

$$g(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$$\triangleq 1_A(x)$$

where $1_A(x)$ is the <u>indicator function</u>

The n^{th} central moment of a r.v. X is

$$E\left[(X - E[X])^n\right] = \int_{-\infty}^{\infty} (x - E[X])^n f_X(x)\, dx$$

$$= \sum_{x_i} (x_i - E[X])^n p_X(x_i)$$

Central moments can be used to describe the "shape" of a distribution.

The 2^{nd} central moment is called the variance of r.v. X and is given by

$$Var[X] = \sigma_X^2 = E\left[(X - E[X])^2\right]$$

The variance of X describes the width (spread) of the distribution of X.

The standard deviation of X is

$$\sigma_X = \sqrt{Var[X]}$$

The expected value of a r.v. $g(X)$ is

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$= \sum_{x_i} g(x_i)\, p_X(x_i) \quad \text{if } X \text{ is discrete}$$

Note: The mean, n^{th} moment, n^{th} central moment are all special cases of $E[g(X)]$

Probability can also be interpreted as an expected value

let $A \subset \mathbb{R}$ and

$$g(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

$$\triangleq 1_A(x) \quad (I_A(x))$$

where $1_A(x)$ is the <u>indicator function</u> on the set A.

We have that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$= \int_{-\infty}^{\infty} 1_A(x) f_X(x) dx$$

$$= \int_A f_X(x) dx$$

$$= Pr(X \in A)$$

<u>Note</u>: If $Y = g(X)$ we have two ways to find μ_Y

$$\mu_Y = E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy$$

$$= E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Similar to

$$\Pr(Y \in A) = \int_A f_Y(y)\, dy$$

$$= \Pr(g(X) \in A) = \int_{x:\, g(x) \in A} f_X(x)\, dx$$

Properties of Mean and Variance :

Let X be a r.v. and $a \in \mathbb{R}$ be a constant.

0) $E[a] = a$

1) $E[aX] = a\, E[X]$

2) $E[g(X) + h(X)] = E[g(X)] + E[h(X)]$

Proof of 0), 1), 2) follow from linearity of integrals

3) $\text{Var}[X] = E[X^2] - (E[X])^2$

<u>Pf:</u>

$$\text{Var}[X] = E[(X - E[X])^2]$$

$$= E[X^2 - 2X\,E[X] + (E[X])^2]$$

$$= E[X^2] - 2\cancel{X}(E[X])^2 + (E[X])^2$$

by linearity

$$= E[X^2] - (E[X])^2$$

4) $Var[aX] = a^2 Var[X]$

Pf: $Var[aX] = E[(aX - E[aX])^2]$

$= E[a^2 (X - E[X])^2]$ } by linearity

$= a^2 E[(X - E[X])^2]$

$= a^2 Var[X]$

5) $Var[X+a] = Var[X]$

Pf:

$Var[X+a] = E[(X+a - E[X+a])^2]$

$= E[(X - E[X])^2]$, by linearity

$= Var[X]$

6) Suppose $X \geq 0$ ($f_X(x) = 0$ for $x < 0$).

Then $E[X] \geq 0$ and $E[X] = 0$

if and only if $X = 0$ ($f_X(x) = \delta(x)$)

Pf:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x f_X(x) dx \geq 0$$

second part left as exercise

7) $E[X^2] \geq 0$, and $E[X^2] = 0$

iff $X = 0$.

$Var[X] \geq 0$, and $Var[X] = 0$ iff $X = \mu_X$

Pf

Replace X by X^2 and $(X - E[X])^2$

in 6)