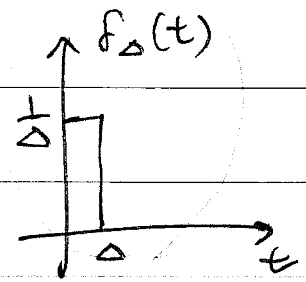
Chap 2. LTI

6/22 ①

1. CT Convolution

1) Derivation of CT convolution



$\rightarrow \boxed{LTI} \rightarrow h_\Delta(t)$

$\lim_{\Delta\to 0} \delta_\Delta(t) = \delta(t)$ ↑ Dirac Delta Func.

$\lim_{\Delta\to 0} h_\Delta(t) = h(t)$ ↑ Impulse Respone.

<Input> <output>

$\delta(t) \longrightarrow \boxed{LTI} \longrightarrow h(t)$

$(x(t) \longrightarrow \boxed{S} \longrightarrow y(t))$

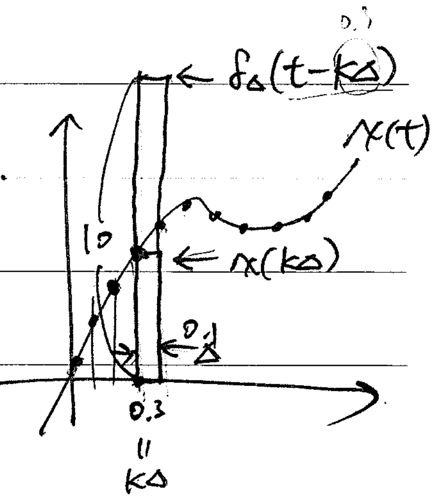
∘ approximate $x(t)$ as :

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \underline{x(k\Delta)\cdot\Delta\cdot\delta_\Delta(t-k\Delta)}$$

$\times\ 0.1 \quad \times\ 10$

$\rightarrow x(k\Delta)$

e.g.



$\Delta = 0.1$

$k=3$

$\frac{1}{\Delta} = 10$

Due to both linearity and time-invariance

→ see Fig. 2.15 on the text.

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\,\delta_\Delta(t-k\Delta) \rightarrow \boxed{\text{LTI}} \rightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\, h_\Delta(t-k\Delta)$$

as $\Delta \to 0$: $\quad \Sigma \to \int$, $\Delta \to d\tau$

$\Rightarrow \quad k\Delta \to \tau \quad$: continuous-valued variable since k goes to infinity.

$\Rightarrow \quad h_\Delta(t-k\Delta) \to h(t-\tau)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \Leftrightarrow y(t) = x(t) * h(t)$$

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_\Delta(t-k\Delta)\,\Delta$$

"Convolution"

In ECE 202, you learned Laplace Transform

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = X(s)\,H(s)$$

* since multiplication is commutative, it follows:

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{L}} Y(s) = H(s)X(s)$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \qquad \qquad ||$$

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = X(s)H(s)$$

So, convolution is commutative.

* conv. also satisfies both distributive & associative property.

Distributive propery has implications w.r.t two LTI systems in parallel:

$$x(t) * \{h_1(t) + h_2(t)\} \xleftrightarrow{\mathcal{L}} X(s)\{H_1(s) + H_2(s)\}$$

$$= x(t)*h_1(t) + x(t)*h_2(t) \qquad = X(s)H_1(s) + X(s)H_2(s)$$

* Two LTI systems in parallel can be replaced by a single LTI system with impulse response $h(t) = h_1(t) + h_2(t)$

* conv. : convolution

* Associative property has implications w.r.t two LTI systems in series:

$$x(t) * h_1(t) * h_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)\, H_1(s)\, H_2(s)$$

$$= (x(t) * h_1(t)) * h_2(t) \qquad = (X(s)\, H_1(s))\, H_2(s)$$

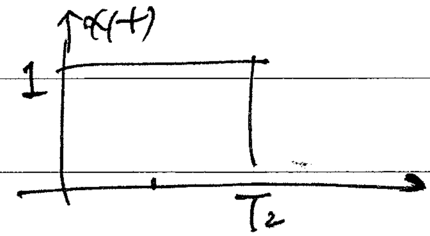
$$= x(t) * (h_1(t) * h_2(t)) \qquad = X(s)\, (H_1(s)\, H_2(s))$$

* Two LTI systems in series can be replaced by a single LTI system with impulse response $h(t) = h_1(t) * h_2(t)$
* Further, commutativity dictates that $h(t) = h_2(t) * h_1(t)$.
  So order doesn't affect overall input / output relationship.
* Conv. is a general mathematical operation, so it doesn't have to be a signal with an impulse response.
  e.g. we can convolve two signals.
* Later, we will show that if you multiply two signals in the time domain, you get convolution in the freq. domain.

- e.g. $y(t) = \int_{t-T_1}^{t} x(\tau)\,d\tau$

input $x(t) = u(t) - u(t-T_2) = \mathrm{rect}\left(\frac{t-\frac{T_2}{2}}{T_2}\right)$

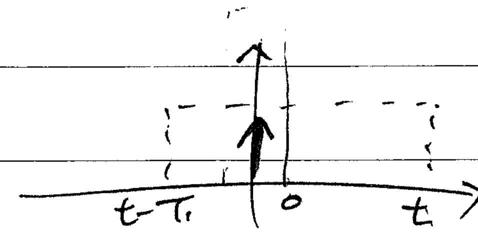
unit step Func.



$T_1 < T_2$

Impulse Response $h(t) = ?$

$\rightarrow h(t) = \int_{t-T_1}^{t} \delta(\tau)\,d\tau$

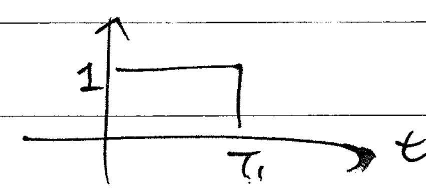


As long as $t-T_1<0$ and $t>0$, capture area of delta func. equal to one.

That is : $h(t) = \begin{cases} 1, & 0<t<T_1 \\ 0, & t<0 \text{ or } t>T_1 \end{cases}$

$\rightarrow \underline{h(t)} = u(t) - u(t-T_1) = \mathrm{rect}\left(\frac{t-\frac{T_1}{2}}{T_1}\right)$

"Impulse response"



It follows that: $(T_1 = \infty)$

$h(t) = u(t)$ is the impulse response of $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$$= x(t) * u(t)$$