Notice that I am using the properties table attached to this exam version.

(15 pts) **1.** Compute the Fourier transform of the DT signal

$$x[n] = n^2u[n-2] - n^2u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$\mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} \left(n^2 u[n-2] - n^2 u[n+2]\right) e^{-j\omega n}$$

$$= \sum_{n=-2}^{2} (-1) n^2 e^{-j\omega n}$$

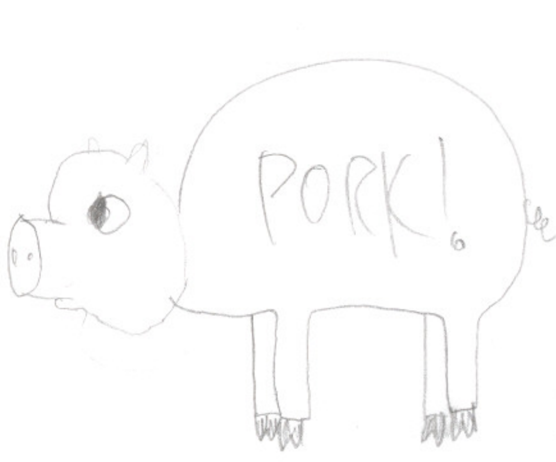
$$= -4e^{+j\omega 2} - 1e^{+j\omega 1} + 0 - 1e^{-j\omega 1} - 4e^{-j\omega 2}$$

$$= -4(e^{j\omega 2} + e^{-j\omega 2}) - (e^{j\omega} + e^{-j\omega})$$

$$= -4(\cos\omega 2 + \cancel{j\sin\omega 2} + \cos\omega 2 - \cancel{j\sin\omega 2})$$

$$- (\cos\omega + \cancel{j\sin\omega} + \cos\omega - \cancel{j\sin\omega})$$

$$\boxed{X(\omega) = -8\cos(2\omega) - 2\cos\omega}$$



(15 pts) **2.** Show that the Fourier transform of the CT signal $x(t) = cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

$$x(t) = \mathcal{F}^{-1}(\mathcal{X}(\omega)) = \frac{1}{2\pi}\cdot\pi\int_{-\infty}^{\infty}(\delta(\omega+\omega_0) + \delta(\omega-\omega_0))e^{j\omega t}\,d\omega$$

$$= \frac{1}{2}(e^{-j\omega_0 t} + e^{j\omega_0 t})$$

$$= \frac{1}{2}(\cos(\omega_0 t) \cancel{- j\sin\omega_0 t} + \cos\omega_0 t + \cancel{j\sin\omega_0 t})$$

$$= \frac{1}{2}(2\cos(\omega_0 t))$$

$$\boxed{\begin{array}{c} \mathcal{F}^{-1}(\mathcal{X}(\omega)) = \cos(\omega_0 t) \\ \downarrow \\ \mathcal{F}(x(t)) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0) \\ \text{Q.E.D.} \end{array}}$$

(15 pts) **3.** Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{cos\omega}$. Explain why Bob's answer is wrong.

x[n] is real and even. This means that $X(\omega) = \mathcal{F}(x[n])$ must be real and even by (39). However, $X(\omega) = \frac{j}{\cos\omega}$ is pure imaginary. Therefore, $\frac{j}{\cos\omega}$ is not correct.

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{sin\omega}$. Could Alice be right? Explain.

As said above, $X(\omega)$ needs to be even. $X(\omega) = \frac{1}{\sin\omega}$ is odd and cannot be correct.

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

$X(\omega) = \frac{1}{\omega^2}$ is not periodic. Since $X(\omega)$ is the transform of a D.T. signal, it has to be periodic. Thus, $X(\omega) = \frac{1}{\omega^2}$ is not correct.

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$Y(\omega)\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right) = 2X(\omega)$$

$$\mathcal{F}^{-1}\left(Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega)\right) = \mathcal{F}^{-1}(2X(\omega))$$

By (24):

$$\mathcal{F}^{-1}(Y(\omega)) - \mathcal{F}^{-1}\left(\tfrac{3}{4}e^{-j\omega}Y(\omega)\right) + \mathcal{F}^{-1}\left(\tfrac{1}{8}e^{-2j\omega}Y(\omega)\right) = 2x[n]$$

By (25):

$$\boxed{y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]}$$

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

$$\mathcal{X}(\omega) = \mathcal{F}(x[n]) = \mathcal{F}\left(\left(\tfrac{1}{4}\right)^n u[n]\right)$$ $\left|\tfrac{1}{4}\right| < 1$, so by (42)...

$$\mathcal{X}(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\mathcal{F}(\text{Output}) = Y(\omega) = H(\omega)\,\mathcal{X}(\omega)$$

$$\boxed{Y(\omega) = \frac{2}{\left(1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}}$$

(15 pts) b) Find the unit impulse response of this system.

$$H(\omega) = \frac{2}{(1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega})}$$

Factor!

$$q = e^{-j\omega}$$

$$(1-\tfrac{3}{4}q+\tfrac{1}{8}q^2)$$

$$\tfrac{1}{8}(8-6q+q^2)$$

$$\tfrac{1}{8}(2-q)(4-q)$$

$$H(\omega) = \frac{16}{(2-e^{-j\omega})(4-e^{-j\omega})}$$

Partial fraction!

$$\frac{16}{(2-q)(4-q)} = \frac{A}{2-q} + \frac{B}{4-q}$$

$$16 = (4-q)A + (2-q)B$$

$$= 4A - Aq + 2B - Bq$$

$$16 = (4A+2B) - (A+B)q$$

$$16 = 4A+2B$$

$$A+B=0 \rightarrow B=-A$$

$$16 = 4A-2A$$

$$16 = 2A$$

$$A = 8$$

$$B = -8$$

$$H(\omega) = \frac{8}{2-e^{-j\omega}} + \frac{-8}{4-e^{-j\omega}}$$

$$= \frac{4}{1-\frac{1}{2}e^{-j\omega}} + \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{h[n] = \mathcal{F}^{-1}(H(\omega)) = 4\cdot\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]}$$

(20 pts) **5.** Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

OK.

$$x(at+b) \xrightarrow{F} \frac{e^{j\omega\frac{b}{a}}}{-a}\mathcal{X}\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$\cancel{\mathcal{F}(x(at+b)) = \int_{-\infty}^{\infty} x(at+b)e^{-j\omega(at+b)}dt}$$

$$\cancel{= e^{-j\omega b}\int_{-\infty}^{\infty} x(at+b)e^{-j\omega at}dt}$$

$t_2 = at + b$

$dt_2 = a\,dt$

$\frac{dt_2}{a} = dt$

If $t = \infty$, $t_2 = -\infty$, since $a < 0$. similarly $t = -\infty \to t_2 = \infty$

$t = \frac{t_2 - b}{a}$

$$\mathcal{F}(x(at+b)) = \int_{t=-\infty}^{\infty} x(at+b)e^{-j\omega t}dt$$

$$= \int_{t_2=\infty}^{-\infty} x(t_2)e^{-j\omega\frac{t_2-b}{a}}\frac{1}{a}dt_2$$

$$= \frac{e^{j\omega\frac{b}{a}}}{-a}\int_{-\infty}^{\infty} x(t_2)e^{-j\frac{\omega}{a}t_2}dt_2$$

$$\boxed{\mathcal{F}(x(at+b)) = \frac{e^{j\omega\frac{b}{a}}}{-a}\mathcal{X}\left(\frac{\omega}{a}\right) \quad \text{Q.E.D.}}$$