

(15 pts) 1. Compute the Fourier transform of the DT signal

$$x[n] = n^2 u[n-2] - n^2 u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$\begin{aligned} x[n] &= n^2 u[n-2] - n^2 u[n+2] \\ &= n^2 (u[n-2] - u[n+2]) \end{aligned}$$

$$X(\omega) = \sum_{n=-2}^{\infty} n^2 e^{-j\omega n}$$

$$= -4e^{j2\omega} - e^{j\omega} + e^{-j\omega} - 4e^{-2j\omega}$$

$$= -2\cos(\omega) - 8\cos(2\omega)$$

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(15 pts) 2. Show that the Fourier transform of the CT signal $x(t) = \cos(\omega_c t)$ is $X(\omega) = \pi\delta(\omega + \omega_c) + \pi\delta(\omega - \omega_c)$.

$$\begin{aligned} X(\omega) &\stackrel{\mathcal{F}}{\rightarrow} \frac{1}{2\pi} \int_{-\infty}^{\infty} [\pi\delta(\omega + \omega_c) + \pi\delta(\omega - \omega_c)] e^{j\omega t} d\omega \\ &= \frac{1}{2} [e^{j\omega_c t} + e^{-j\omega_c t}] \\ &= \boxed{\cos(\omega_c t)} \end{aligned}$$

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$$x[-n] = \frac{1}{g[-n]} = \frac{1}{(g[n])^*} = \frac{1}{g[n]^*}$$

$X(\omega)$ is real & even

{(5 pts) 3. Given is a DT signal $x[n] = \frac{1}{g[n]}$ where $g[n]$ is a pure imaginary signal and an odd function of n .

a) Bob claims that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\cos \omega}$. Explain why Bob's answer is wrong.

5 $X(\omega)$ is real & even
 $X(\omega)$ is real & even so $X(\omega)$ must be real & even
 \therefore Bob is wrong

b) Alice says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\sin \omega}$. Could Alice be right? Explain.

5 $X(\omega)$ is real & even so $X(\omega)$ must be real & even
 $X(\omega)$ is real & odd
 \therefore Alice is wrong

c) Devin says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\omega}$. Could Devin be right? Explain.

5 Devin is wrong because the F.T. of any D.T. signal is periodic.

Neat and clean!

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

14 $H(\omega) = \frac{Y(\omega)}{X(\omega)}$

$$Y(\omega) = H(\omega)X(\omega) = \frac{2X(\omega)}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$Y(\omega) \left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right) = 2X(\omega)$$

$$Y(\omega) = \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

taking IFT

$$\therefore \boxed{y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]}$$

and using
time shifting
property

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{3}\right)^n u[n]$?

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{j\omega}}$$

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$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{2}{1 - \frac{1}{2}e^{j\omega}} \cdot \frac{1}{1 - \frac{1}{3}e^{j\omega}}$$

$$= \frac{16}{(e^{j\omega} - 2)(e^{j\omega} - 4)} \cdot \frac{4}{4 - e^{j\omega}}$$

$$= \frac{16}{(e^{j\omega} - 2)(e^{j\omega} - 4)} \cdot \frac{-4}{(e^{j\omega} - 4)}$$

$$= \frac{64}{(e^{j\omega} - 2)(e^{j\omega} - 4)^2}$$

can
stop
here

(15 pts) b) Find the unit impulse response of this system.

$$H(\omega) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$h[n] = \mathcal{Z}^{-1}[H(\omega)]$$

$$= \frac{16}{(e^{j\omega} - 2)(e^{j\omega} - 4)}$$

$$A(x-2) + B(x-4) = 16$$

$$x=2, -2B = 16, B = -8$$

$$x=4, 2A = 16, A = 8$$

$$= \frac{16}{(e^{j\omega} - 2)(e^{j\omega} - 4)}$$

$$= \frac{-8}{(e^{j\omega} - 2)} + \frac{8}{(e^{j\omega} - 4)}$$

$$= \frac{-2}{(1 - \frac{3}{4}e^{j\omega})} + \frac{4}{(1 - \frac{1}{2}e^{j\omega})}$$

$$h[n] = \boxed{-2 \cdot \left(\frac{3}{4}\right)^n u[n] + 4 \left(\frac{1}{2}\right)^n u[n]}$$

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(20 pts) 5. Use the definition of the Fourier transform (not the properties listed in the table) to prove the following Fourier transform property

$$x(at+b) \xrightarrow{F} \frac{e^{j\omega b}}{-a} X\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$\mathcal{F}\{x(at+b)\} = \int_{-\infty}^{\infty} x(at+b) e^{-j\omega t} dt \qquad \frac{e^{j\omega b}}{-a} X\left(\frac{\omega}{a}\right)$$

$$\begin{aligned} \text{Let } \tau &= at+b \\ d\tau &= a dt \\ dt &= \frac{d\tau}{a} \\ t &= \frac{\tau-b}{a} \end{aligned}$$

$$\begin{aligned} t = -\infty, \tau &= -\infty \\ t = \infty, \tau &= \infty \end{aligned}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau-b}{a}} d\tau$$

$$= \frac{e^{j\omega \frac{\tau-b}{a}}}{-a} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{a}} d\tau$$

! Flip bounds of integration

$$= \boxed{\frac{e^{j\omega \frac{b}{a}}}{-a} X\left(\frac{\omega}{a}\right)}$$

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