DTFT

July 28

<u>Ex</u>

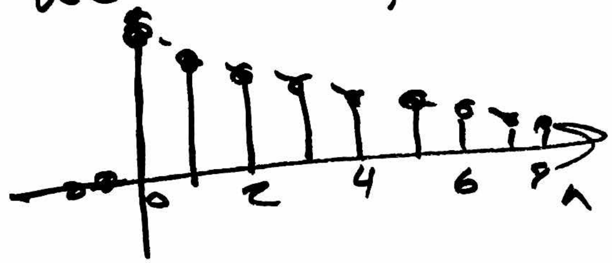
$$x[n] = \alpha^n u[n] \qquad |\alpha| < 1$$

Find $X(\omega)$:

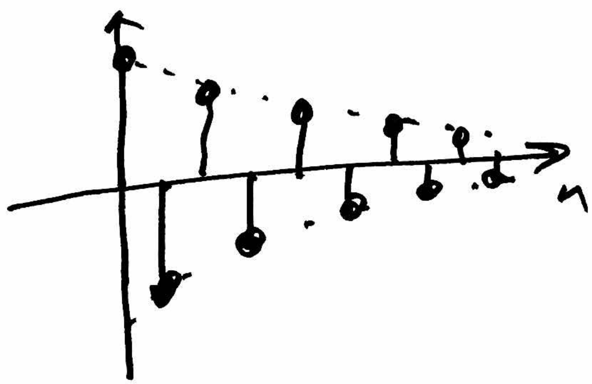
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

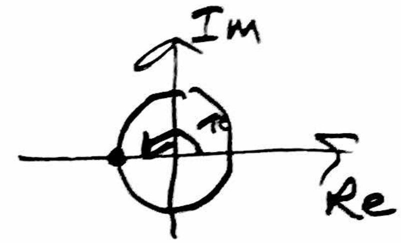
What if we had a filter $h_1[n] = \alpha^n u[n]$, $0<\alpha<1$?



$$h_2[n] = (-\alpha)^n u[n]$$



How are $H_2(\omega)$ and $H_1(\omega)$ related?



$$h_2[n] = (-\alpha)^n u[n] \qquad -1 = e^{j\pi}$$

$$= (e^{j\pi}\alpha)^n u[n]$$

$$= e^{j\pi n}\alpha^n u[n]$$

$$= e^{j\pi n} h_1[n]$$

frequency shift $\quad e^{j\omega_0 n} x[n] \xleftrightarrow{\text{DTFT}} X(\omega - \omega_0)$

$$H_2(\omega) = H_1(\omega - \pi)$$

Ex $x[n] = e^{j\omega_0 n}$ find $X(\omega)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-j(\omega-\omega_0)n}$$

The infinite sum doesn't directly converge because $|e^{-j(\omega-\omega_0)n}| = 1$.

One approach is ~~a limiting to~~ to use a limit

$$\tilde{x}_N[n] = \begin{cases} e^{j\omega_0 n} & ; |n| < N \\ 0 & , \text{else} \end{cases}$$

$$\tilde{X}_N(\omega) = \sum_{n=-N}^{N} e^{-j(\omega-\omega_0)n}$$

$$X(\omega) = \lim_{N\to\infty} \tilde{X}_N(\omega)$$

Or, we can look at the synthesis (Inverse DTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$= e^{j\omega_0 n} \quad \Rightarrow \text{guess } X(\omega) = 2\pi\,\delta(\omega-\omega_0)$$

verify:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega-\omega_0) e^{j\omega n} d\omega$$

$$= e^{j\omega_0 n}\int_{-\pi}^{\pi} \delta(\omega-\omega_0)\, d\omega$$

$$= e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\,\delta(\omega-\omega_0) \qquad -\pi < \omega < \pi$$

$$= 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega-\omega_0-2\pi l)$$

$$= \text{rep}_{2\pi}\{2\pi\,\delta(\omega-\omega_0)\}$$

Ex DTFT of a sampled signal

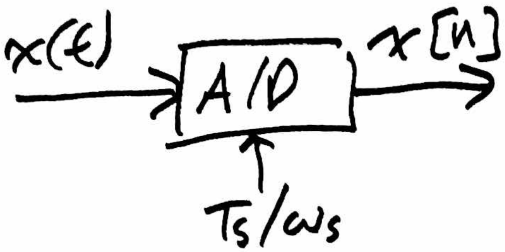
$x(t) = \cos(200\pi t)$

Find $X(\omega)$

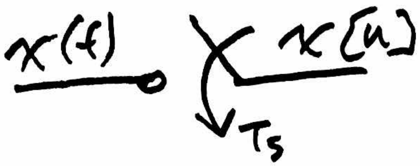
First, we need to sample $x(t)$.

$\omega_N = 400\pi \Rightarrow$ choose $\omega_s = 800\pi$

$\omega_s = \frac{2\pi}{T_s} \quad T_s = \frac{2\pi}{\omega_s}$



$$x[n] = x(t)\Big|_{t = T_s n}$$

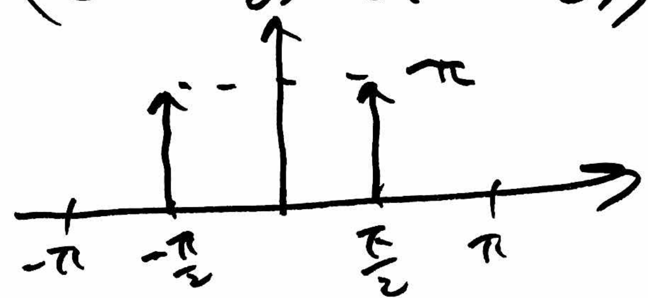


$$= x\left(\frac{1}{400} n\right) = \cos\left(200\pi \cdot \frac{1}{400} n\right)$$

$$= \cos\left(\frac{\pi}{2} n\right)$$

$$= \frac{1}{2} e^{j\frac{\pi}{2} n} + \frac{1}{2} e^{-j\frac{\pi}{2} n}$$

$$X(\omega) = \pi\left(\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right)\right) \quad -\pi < \omega < \pi$$



<u>Dirac deltas</u>

$x_2(t) = \cos(500\pi t)$ $\qquad \omega_N = 1000\pi$

$\qquad\qquad\qquad\qquad\qquad\quad \omega_s = 2000\pi$

$$x_2[n] = \cos(500\pi t)\Big|_{t=nT_s} = \cos\left(500\pi \cdot \tfrac{1}{1000} n\right)$$

$$= \cos\left(\tfrac{\pi}{2} n\right) \Rightarrow \text{same as the previous example}$$

~~Ex~~